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Resolution attributes for geophysical inversion models: Depth of investigation and novel measures

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ABSTRACT

Assessing the reliability of inversion models derived from geophysical measurements is crucial for a reliable interpretation. An interpretation depends critically on the interpreter being able to discern between the characteristics of the inversion model that can be trusted as more or less well resolved and the ones that are more dubious. This paper analyses the resolution measure 'depth of investigation' from a conceptual and a computational viewpoint and proposes two definitions that incorporate all aspects of the inversion and that are free of a user-defined ad hoc parameter. Two more resolution attributes are introduced: a qualified depth of investigation and the depth of required structure . The first one answers the question: What is the minimum depth to a homogeneous halfspace with an interpreter-defined conductivity that will not increase the data residual more than a certain amount? The second one is an 'unqualified' depth of investigation that addresses the question: What is the minimum depth to a homogeneous halfspace with any conductivity that will not increase the data residual more than a certain amount? This latter measure indicates the depth below which no structure is needed to fit the data. Finally, measures are defined that will provide estimates of the vertical resolution width as a function of depth. All of the resolution measures presented in this paper are based on the posterior model resolution matrix.

Key words: Inversion, Interpretation, Uncertainty.

INTRODUCTION

Quite a number of ways of arriving at a depth of investigation (DOI) measure have been suggested in the published literature – and more have probably been used, but never published. The quest for a reasonable measure of the DOI has been on for as long as geophysical methods have been applied. Quite obviously, it is clear that we cannot probe the subsurface down to the centre of the Earth with any geophysical method applied on or above the ground, and the question naturally arises: How deep does the method look? To which depth can I trust the model resulting from inversion of the data? Throughout this paper, I will give specific attention to electromagnetic methods (EM), in particular one-dimensional (1D) inversion models, but most of the results of the conceptual analyses will have general validity also in other situations.

The concepts to be used in the following analysis of DOI measures are as follows:

- What is the defining principle?
- Based on which parameter is the DOI defined?
- Does the definition require an ad hoc, user-defined, limiting value?
- Is the method prior or posterior?

Ideally, the defining principle must be intuitively clear and reasonable, and preferably the DOI definition should have no need for an ad hoc, user-defined limiting value for the defining parameter because this introduces an element of subjectivity and arbitrariness. Furthermore, it should be posterior,

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meaning that all covariances between model parameters are taken into account in the DOI measure, and, finally, it should be calculable based on term already available from an inversion of the data. The next paragraph will elaborate on these criteria.

All DOI measures are based on a defining principle and most often a defining parameter. In most DOI measures, the DOI is determined as the depth where the defining parameter passes a limiting, user-defined, threshold value. Furthermore, almost all DOI measures can be categorized as either prior, meaning that the defining parameters and their limiting values are defined based on parameter sensitivities, that is the Jacobian of the inversion – or posterior, meaning that an inversion must take place to provide the parameter on which the DOI is defined. It has been shown (Christensen and Lawrie, 2012, 2014; Smiarowski and Mulè, 2014) that a prior analysis of resolution is inferior to a posterior analysis because it does not take the coupling between model parameters into account; it does not reflect the covariances between the model parameters and thereby neglects important characteristics of the geophysical method in question. Faulty conclusions about the assets and drawbacks of a given method can be reached when relying exclusively on a prior analysis, and, in general, it would therefore be wise to avoid DOI measures based on a prior analysis if other measures are available. If a limiting parameter is needed for the definition of the DOI, the developer or user will have to decide 'what looks the best'/'what seems reasonable' which obviously introduces a subjective choice, whether well informed or not. Finally, calculating the DOI measure should not be computationally heavy. However, the more important and the more desirable an attribute is, the more we are willing to – and should – make concessions on this demand.

One of the first attempts of defining a DOI was published in Roy and Apparao (1971). They studied the sensitivity functions for a homogeneous halfspace of DC geoelectrical methods for a series of electrode configurations and defined the DOI as the depth where the sensitivity function reached its maximum. Data and data errors were not considered, and rather than calling this a DOI in the modern sense of the concept, I would call this a 'characteristic depth parameter' for the method and configuration. Since then, quite often the depth where the normalized integrated sensitivity function reached a value of one half, that is where there were equal amounts of sensitivity above and below, has been used as a pseudo-depth parameter for imaging measured data. Still, I would also not characterize this definition as a DOI.

Spies (1989) outlines a method of estimating the DOI for electromagnetic soundings in both frequency and time domain based on the signal-to-noise level of the data with the deepest depth penetration (low frequencies or late times) and the integrated conductivity of the model in question. The DOI is the diffusion depth for the last data point with a specific signal/noise ratio. The paper introduces one of the important aspect of a DOI estimate: that it depends on the signal-to-noise level of the data – plus, of course, the subsurface model and the method applied. The limitation of the approach lies in the assumption that the diffusion depth depends on the subsurface model in a simple way: that a layered model can be substituted with a homogeneous halfspace with the mean conductivity. This is of course an approximation, but for inductive EM methods it is actually not a bad one (Christensen, 2016). The defining principle is tied to the concept of diffusion depth, and the ad hoc parameter for this definition is the signal/noise ratio that defines the last useful data point.

Many of the definitions of DOI are based on the sensitivities, that is the derivatives of the response with regard to the model parameters. A recent variation on this principle is given in Christiansen and Auken (2012). The method is based on the diagonal elements of the noise-normalized Jacobian matrix: $[J^T \cdot C_e^{-1} \cdot J]$, where J is the Jacobian and C_e is the data error covariance matrix of the final inversion model. In a least-squares iterative inversion scheme, this matrix is always formed. The DOI is defined as the model depth where the cumulated sum of diagonal elements, starting from the bottom of the model, reaches a certain limiting value. Being based on the noise normalized sensitivities, and as such it is a prior measure and it requires an ad hoc limiting value for the sum of the diagonal elements. However, the fact that the analysis is performed on the final inversion model places it somewhere between a prior and a posterior analysis and makes it quite sound; certainly better than several other approaches. The method can be used in inversion with multi-layer models. Providing a DOI for few-layer models requires that they be reformulated/approximated by a multi-layer model.

The anteroposterior aspect of the method of Christiansen *et al.* could, however, be changed into a posterior approach quite easily by using the posterior equivalent of the sensitivities, namely the inverse of the diagonal elements of the posterior covariance matrix: $1/\mathrm{var}_i$, where var_i is the posterior variance of the ith model parameter. However, it would still require an ad hoc limiting value to be defined for the cumulative sum of the diagonal elements.

An approach which is posterior in character is based on the fact that inversion results with a multi-layer model depend on the initial/prior model. To define the DOI, inversions are carried out with a resistive and a conductive prior model, and the DOI is defined as the depth where the two inversion results differ more than a certain limiting value. This methodology provides a reliable, posterior approach to defining the DOI and is intuitively clear, but it requires two inversion runs to be carried out, and there is still an ad hoc limiting value for the model difference. The approach has been suggested by several authors; see, for example, Oldenburg and Li (1999). Somewhat outside the scope of this paper, but related to the previous approach, in reversible-jump Monte Carlo methods (Hawkins *et al.*, 2018), a DOI can be defined as the depth where the posterior model distribution deviates from the assumed prior model distribution more than a certain user-defined amount. However, Markov chain Monte Carlo methods, at least in their original form, are computationally much more expensive than standard deterministic inversion which forms the basis of my suggested new definitions of a DOI measure. A recent comparison of different DOI definitions can be found in Asch *et al.* (2015), and Flores *et al.* (2013) investigated a DOI definition for transient EM soundings.

A NEW DEFINITION OF DEPTH OF INVESTIGATION

In this section, I will present a novel definition of the DOI that fulfils the four ideal criteria laid out in the Introduction: the defining principle is intuitively clear and reasonable; the method is posterior and parameter-free, and it is straightforward to calculate without being computationally heavy. The method can be used in multi-layer inversion. The suggested DOI measure is based on the posterior model resolution matrix (MRM) so first I will present the inversion approach and the properties of the MRM.

One-dimensional inversion methodology

The 1D inversion formulation on which the calculation of the MRM is based is an iterative damped least squares approach (Menke, 1989) with models consisting of horizontal, homogeneous and isotropic layers. At the *n*th iteration, the updated model is given by:

$$\mathbf{m}_{n+1} = \mathbf{m}_n + \left[\mathbf{G}_n^T \mathbf{C}_e^{-1} \mathbf{G}_n + \frac{1}{\sigma_m^2} \mathbf{C}_m^{-1}\right]^{-1} \cdot \left[\mathbf{G}_n^T \mathbf{C}_e^{-1} (\mathbf{d}_{\text{obs}} - \mathbf{g}(\mathbf{m}_n))\right] \quad (1)$$

where $\mathbf{m}$ is the model vector containing the logarithm of the model parameters, $\mathbf{G}$ is the Jacobian matrix containing the derivatives of the data with respect to the model parameters, T indicates matrix transpose, $\mathbf{C}_e$ is the data error covariance matrix, $\mathbf{C}_m$ is a model covariance matrix imposing vertical smoothness constraints on the multi-layer model, σ_m is the standard deviation assigned to the model covariance matrix determining the strength of the smoothness constraint, $\mathbf{d}_{\text{obs}}$ is the field data vector and $g(\mathbf{m}_n)$ is the nonlinear forward response vector of the *n*th model.

A model parameter uncertainty estimate can be derived from a linear approximation to the posterior covariance matrix, $\mathbf{C}_{\text{est}}$, given by:

$$\mathbf{C}_{\text{est}} = \left[\mathbf{G}^T \mathbf{C}_{\text{obs}}^{-1} \mathbf{G} + \frac{1}{\sigma_m^2} \mathbf{C}_m^{-1}\right]^{-1}, \quad (2)$$

where $\mathbf{G}$ is based on the model resulting from the last iteration. The uncertainty estimate is expressed through the standard deviations of the model parameters obtained as the square root of the diagonal elements of $\mathbf{C}_{\text{est}}$ (e.g., Inman *et al.*, 1975).

The model resolution matrix

In essence, the MRM, in this section called $\mathbf{R}$, is a matrix that defines a linear mapping from the true model parameters – and no one knows what they are – onto the model parameters of the inversion model, that is:

$$\mathbf{m}_{\text{est}} = \mathbf{R} \cdot \mathbf{m}_{\text{true}}, \quad (3)$$

where $\mathbf{m}_{\text{est}}$ is the inversion model vector and $\mathbf{m}_{\text{true}}$ is the unknown true model vector.

The MRM is a square, in general non-symmetric, $M \times M$ matrix, where M is the number of multi-layer model parameters. From (3), it is seen that the elements of the *i*th row of the MRM contain the weights with which the true model parameters are multiplied to give the inversion parameter of the *i*th layer, and, in the following, the rows will sometimes be referred to as resolution kernels. If we have perfect resolution, all elements of the row are zero except the element in the *i*th column which is equal to 1, meaning that only the true parameter of the *i*th layer contributes to the parameter value of the *i*th layer of the inversion model, and the MRM becomes the identity matrix. If we have a reasonably good resolution of the *i*th parameter, the *i*th row of the MRM will be a fairly narrow bell-shaped distribution around column number *i*, meaning that the true parameters in a certain depth range around the *i*th layer contribute to the value of the *i*th layer of the inversion model. The width of the distribution indicates an averaging width. A poor resolution is reflected in a resolution kernel with a wide averaging width, meaning that many of the true parameters around the ith layer contribute to the parameter of the ith layer of the inversion model.

The meaning of the columns of the MRM is the reverse of the above. The jth column of the MRM contains the factors with which the jth true parameter contributes to all the parameters of the inversion models, that is the way the jth true parameter spreads its influence over the inversion model parameters.

The expression for the MRM is given as (Menke, 1989):

$$\mathbf{R} = \mathbf{G}^{-g}\mathbf{G}, \tag{4}$$

where $\mathbf{G}$ is the Jacobian matrix and $\mathbf{G}^{-g}$ is the generalized inverse of $\mathbf{G}$. In the standard inversion formulation where data noise is included, we find:

$$\mathbf{R} = \left(\mathbf{G}^T\mathbf{C}_e^{-1}\mathbf{G}\right)^{-1}\mathbf{G}^T\mathbf{C}_e^{-1}\cdot\mathbf{G}. \tag{5}$$

The formulation of equation (5) refers to an unconstrained inversion, and it results in $\mathbf{R} = \mathbf{I}$, that is the MRM is the identity matrix and resolution is perfect. This is the situation in unconstrained few-layer models. However, in the case of a constrained inversion with multi-layer models the expression becomes:

$$\mathbf{R} = \left[\mathbf{G}^T\mathbf{C}_{\mathrm{obs}}^{-1}\mathbf{G} + \mathbf{C}_m^{-1}\right]^{-1}\mathbf{G}^T\mathbf{C}_{\mathrm{obs}}^{-1}\mathbf{G}, \tag{6}$$

where $\mathbf{C}_m$ is the model error covariance matrix, and in this case, $\mathbf{R} \neq \mathbf{I}$, that is resolution is not perfect. In the case where a prior model, C_p, is included in the inversion – rarely so with multi-layer inversion – it will appear in the same way as the model covariance matrix, that is $C_m^{-1} \to C_p^{-1} + C_m^{-1}$ in equation (6).

In practical terms, $\mathbf{R}$ is calculated from the posterior covariance matrix: $[\mathbf{G}^T\mathbf{C}_{\mathrm{obs}}^{-1}\mathbf{G} + \mathbf{C}_m^{-1}]^{-1}$, which is calculated anyway in the inversion, and the model covariance matrix $\mathbf{C}_m$. The derivations are:

$$\mathbf{R} = \left[\mathbf{G}^T\mathbf{C}_{\mathrm{obs}}^{-1}\mathbf{G} + \mathbf{C}_m^{-1}\right]^{-1}\mathbf{G}^T\mathbf{C}_{\mathrm{obs}}^{-1}\mathbf{G} \Rightarrow$$

$$\mathbf{R} = \left[\mathbf{G}^T\mathbf{C}_{\mathrm{obs}}^{-1}\mathbf{G} + \mathbf{C}_m^{-1}\right]^{-1}\cdot\left\{\left[\left(\mathbf{G}^T\mathbf{C}_{\mathrm{obs}}^{-1}\mathbf{G} + \mathbf{C}_m^{-1}\right)^{-1}\right]^{-1} - \mathbf{C}_m^{-1}\right\}$$

$$\mathbf{R} = \mathbf{I} - \left[\mathbf{G}^T\mathbf{C}_{\mathrm{obs}}^{-1}\mathbf{G} + \mathbf{C}_m^{-1}\right]^{-1}\cdot\mathbf{C}_m^{-1} = \mathbf{I} - \mathbf{C}_{\mathrm{est}}\cdot\mathbf{C}_m^{-1}, \tag{7}$$

where $\mathbf{C}_{\mathrm{est}}$ is the posterior covariance matrix. All matrices are calculated during the inversion process, so it is straightforward – and computationally inexpensive – to calculate the MRM for the final inversion model and subsequently perform the analysis of the MRM that will produce the DOI.

It is important to notice that the MRM, on which the suggested DOI will be based, depends not only on the posterior covariance matrix, and thereby on the data signal-to-noise ratio plus the derivatives of the response with regard to the model parameters, but also on the regularization term expressed through the model covariance matrix $\mathbf{C}_m$. This means that the DOI is influenced by the vertical (and lateral) constraints that are imposed on the inversion. This is a desirable quality. Smoothness constraints introduce additional information and additional demands on the inversion, and the DOI should of course reflect that.

Recalling equation (7), it is seen that the MRM has the following properties:

1. If data error increases, $\mathbf{C}_{\mathrm{obs}}^{-1}$ decreases, or if regularization strength increases, $\mathbf{C}_m^{-1}$ increases, and $\mathbf{C}_{\mathrm{est}}$ becomes more dominated by $\mathbf{C}_m^{-1}$, meaning that $\mathbf{C}_{\mathrm{est}}\cdot\mathbf{C}_m^{-1}$ becomes close to $\mathbf{I}$ and $\mathbf{R} \to \mathbf{0}$. This means that, in the limit, there is no resolution: $\mathbf{m}_{\mathrm{true}}$ does not influence $\mathbf{m}_{\mathrm{est}}$ and $\mathbf{m}_{\mathrm{est}}$ becomes determined by prior information. The column sums of $\mathbf{R}$ will go to zero.

2. If data error decreases, $\mathbf{C}_{\mathrm{obs}}^{-1}$ increases, or if regularization strength decreases, $\mathbf{C}_m^{-1}$ decreases, and $\mathbf{C}_{\mathrm{est}}$ becomes more dominated by $\mathbf{G}^T\mathbf{C}_{\mathrm{obs}}^{-1}\mathbf{G}$ and eventually $\mathbf{C}_{\mathrm{est}}$ will have very small elements, i.e. very small parameter uncertainties and $\mathbf{C}_{\mathrm{est}}\cdot\mathbf{C}_m^{-1} \to \mathbf{0}$ and $\mathbf{R} \to \mathbf{I}$. The prior model will have no influence and the column sums of $\mathbf{R}$ will go to unity.

In the following, it is assumed that inversion is done with a multi-layer model. This is the only situation in which the DOI is defined in a nontrivial way. If the original inversion is done with a few-layer model, a multi-layer model would have to be constructed that as faithfully as possible reproduces the subsurface conductivity structure of the final few-layer model inversion model, and in addition a model covariance matrix for the multi-layer model would have to be defined.

Illustrating the model resolution matrix

Figure 1 presents a plot of an MRM from an inversion of a typical time domain airborne electromagnetic method data set. Figure 1(a) shows the MRM as a colour-coded map of the matrix in an asinh-transformed colour scale, which emphasizes the smaller elements, and Figure 1(b) illustrates the same MRM in a different way by plotting the values (normalized with the maximum value of the row) for every row of the MRM.

In Figure 1(a), it is seen that the contribution from the true layer parameters to the inverted parameter of the top 8–10 layers is distributed fairly evenly, and this is confirmed when considering Figure 1(b) where the values of the top rows are unfocused over this depth interval. The implication is that the inverted parameter is an average of the true parameters over a wide range, and resolution is low.

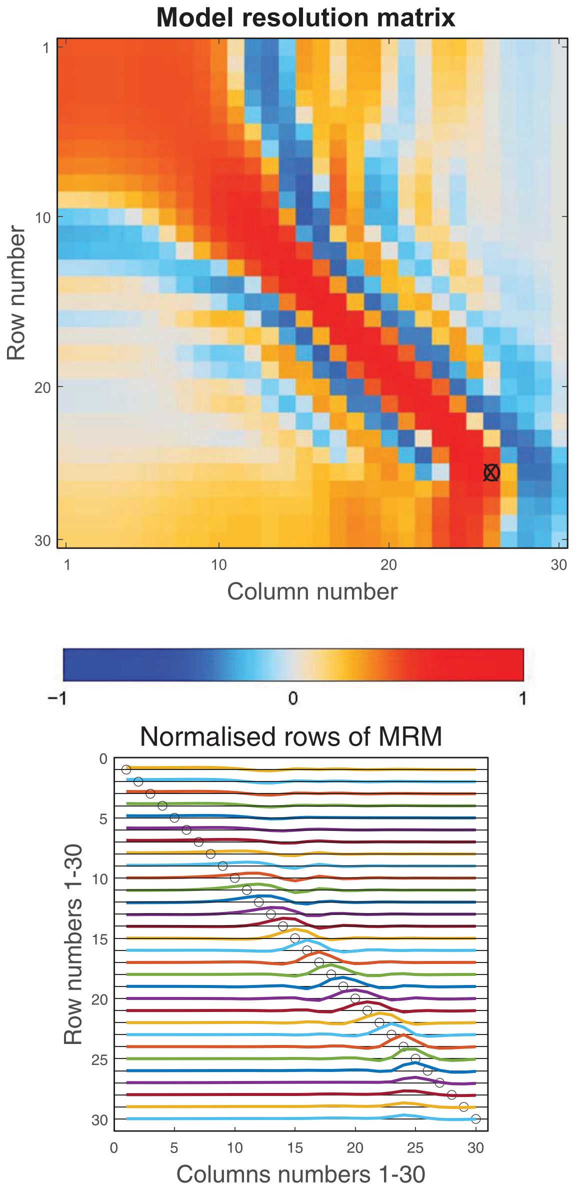

Figure 1 (a): Colour-coded plot of a typical MRM. For clarification, the MRM values are normalized and subjected to an asinh transformation so that the small elements are emphasized. Red indicates high (positive) values and blue small (negative) values. The maximum depth to the maximum element of any row is indicated with a cross in a circle in the diagonal for layer number 26. (b) Plot of the curves of the 30 row values of the same MRM. The black circles indicate the main diagonal where row and column numbers are the same.

In the middle part of the MRM, both Figures 1(a) and 1(b) indicate that the maxima of the resolution kernels follow the layer numbers, meaning that it is the true model parameter of the layers that contribute most to the inverted layer parameter. This is of course the desirable situation and indicates a well-positioned resolution. In the bottom rows, the maximum remains at column ∼ 26 (indicated with a cross in a circle in Fig. 1), meaning that it is the true parameter of the ∼26th layer that contributes most to the model layers below number ∼26 thus indicating that resolution does not penetrate to the bottom of the model.

Figure 1(b) also illustrates that the width of the depth interval contributing most to an inverted value changes with depth, the width being the largest in the upper and lower parts of the model where resolution is the poorest.

Two novel DOI measures derived from the model resolution matrix

Each row of the MRM contains the resolution kernel for every layer of the inverted model containing the weight factors of the true model parameters for that layer. The first DOI measure suggested is therefore the following:

The DOI is the maximum depth of the maximum value of any of the resolution kernels.

This seems intuitively valid. If none of the resolution kernels have their maximum deeper than a certain depth, then the main contributions of the true parameters to the inverted parameter for layers below come from layers above that layer. We have reached the DOI. For the case shown in Figure 1, the DOI thus defined would be placed in layer 26.

Another possible DOI definition that has the same intuitive feeling is to define the DOI as DOI is the maximum depth of the centroid of any of the resolution kernels.

Instead of looking at the maximum value, we now look at the centre-of-mass of the absolute values of the resolution kernels, that is a weighted expression for the maximum, see also the Appendix . Compared with the previous measure, this one is formed through an averaging process and not a point value. So, which one to choose? Well, it could be regarded as an interpreter option, but it could also be argued that the DOI defined by the maximum of the centroid depths – being defined as an average measure and thereby less likely to attain more extreme values – it is a more robust measure.

The Appendix shows the computational issues involved in the calculation of the DOI measures, and the important question whether the centroid measure should be weighted with layer thicknesses or layer numbers is discussed. The conclusion is that a weighting with layer numbers best reflects the nature of the MRM, and the centroid DOI measure plotted in Figure 2 was derived with this weighting. The Appendix also shows detailed analyses of the two DOIs for two models: a fairly resistive one and a fairly conductive one.

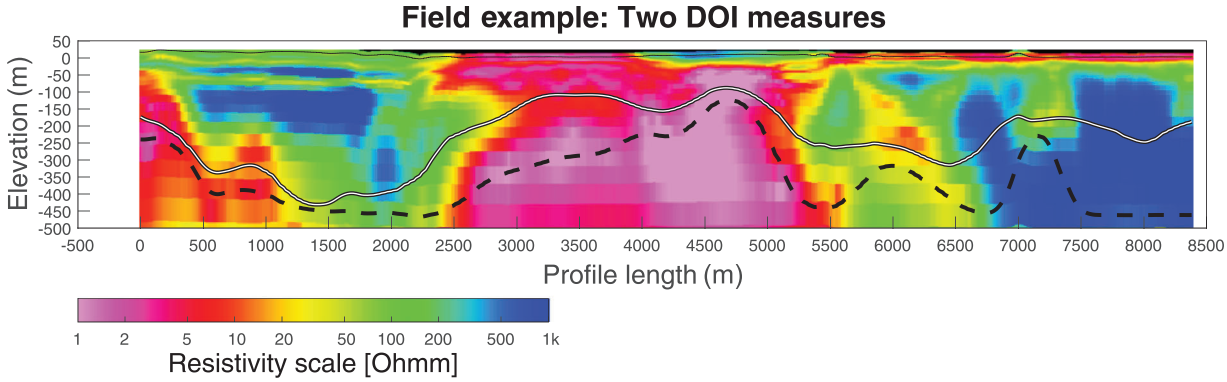

Figure 2 A model section with the two DOI definitions presented in the text. The DOI defined as the maximum depth of the centroid is indicated as a white line with black rims. The DOI defined as the maximum depth to the maximum element of any row is indicated with a dashed line. In this case, the former is everywhere the more shallow of the two.

Application of the DOI measures on field data

To illustrate the resolution attributes presented in this paper, I will present a flight line from an investigation in Northern Territory, Australia, collected with the SkyTEM system in 2017 (Sørensen and Auken, 2004). Data are published and freely available from the web site of Geoscience Australia and can be downloaded from Geoscience Australia's web site https://www.ga.gov.au/. The survey was conducted to provide geophysical data that would assist in a detailed mapping of the groundwater resource in the area. The line is chosen as it contains both conductive and resistive features and lateral gradients that permit a demonstration of how the conductivity regime affects the various resolution measures presented in this paper.

Figure 2 shows a model section of the selected flight line including the two DOI definitions mentioned above. The plot shows that the DOI is more shallow in the more conductive areas and deeper in the more resistive ones. This is what should be expected: the high conductivity areas shield the lower parts of the model, and this fact is reflected in the variation of the DOIs along the flight line. Figure 2 also shows that, on average, and for this model section, the DOI defined by the maximum centroid depth is slightly shallower than the DOI defined by the maximum of the resolution kernel maximum values. Experience shows that, in general, this is the case more often than not, though the two DOI definitions are indeed quite similar.

Measures of resolution width

As can be seen in Figure 1 and from the text of the previous sections, the width of the resolution kernel changes with depth. For every model layer, the width of the resolution kernel expresses the depth interval over which the true model parameters are averaged to form the inverted model parameters, and it is thereby a useful measure of the resolution capacity as a function of depth. Plotting a model section, not with the layer conductivities, but with the widths of the resolution kernels for the layers, provides an analysis section giving immediate visual insight into the averaging depth interval for the inverted parameter (Fig. 3).

I will discuss two different measures of the width of the resolution kernel around the centroid. One is twice the value of the second moment around the centroid, that is twice the standard deviation of the resolution kernel, and as such it is a L2 measure of width. The other is defined as the half-width of the resolution kernel, that is the interval that contains the second and third quartiles of the absolute value of the kernel, so it is a L1 measure. For more detail, see the Appendix . Both measures show the same behaviour of the resolution width as a function of depth though, in general, the L2 measure is greater than the L1 measure. Again, which one to choose can be an interpreter option. This author prefers the half-width defined by the L1 measure. Experience shows that it has a larger dynamic range than the L2 measure, thereby offering a clearer image of the variation with depth of the averaging involved in the resolution. Figure 3 shows a model section demonstrating the DOI measures along the same flight line as in Figure 2. The section illustrates the L1 measure of resolution width as a function of depth in units of layer thickness at that depth. For comparison, Figure 3 also shows the centroid DOI. It is seen that the variations of the DOI along the profile correlate with the resolution width, and comparing Figure 3 with Figure 2 reveals that the half-width reflects both conductivity

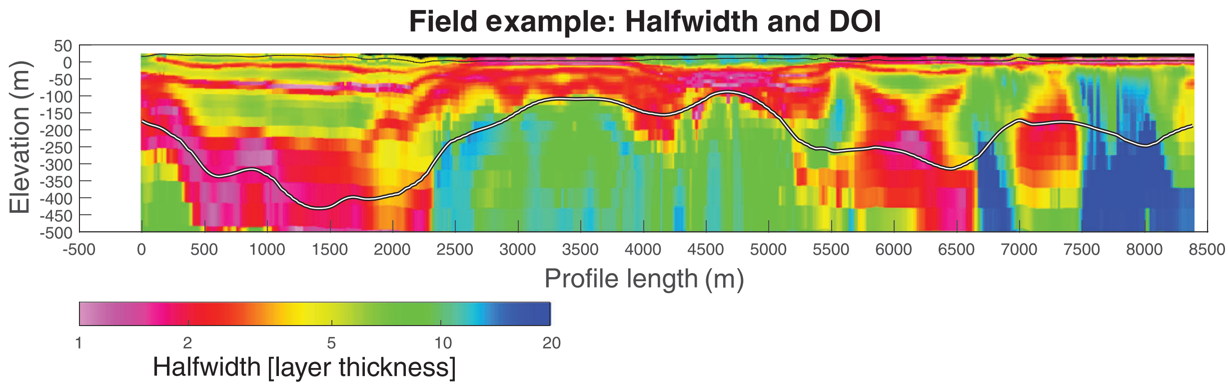

Figure 3 A section showing the resolution width in terms of the half-width of the MRM kernel as defined using the L1 norm for same model section as in Figure 2. The unit of the half-width at a certain depth is the layers thickness at that depth. For comparison, also the DOI based on the maximum depth of the centroid depths is plotted in the section as a white line with black rims.

and depth: A small half-width corresponds to a high conductivity and/or a shallow depth and vice versa.

A QUALIFIED DEPTH OF INVESTIGATION

Defining the qualified depth of investigation

Inspired by the imperfections of most DOI measures, I would like to introduce what I would call a qualified depth of investigation (QDOI). Unlike the majority of the DOI measures, the QDOI answers a specific question:

Given a specific conductivity, what is the minimum depth to a homogeneous halfspace with that conductivity given the condition that the data residual must not increase more than a certain amount.

The QDOI is defined in relation to a specific interpreter-supplied bottom halfspace conductivity. It is the minimum depth for which the part of the model below the QDOI can be substituted with a homogeneous halfspace of that particular conductivity without increasing the data residual more than a certain user-defined factor. (Ups, there it is again … it seems hard to avoid these ad hoc parameters …) It is constructed by sequentially substituting the inversion model parameters from the bottom of the model and up, one by one, with the interpreter-defined conductivity, keeping the rest of the inversion model parameters above the halfspace unchanged. For every step, a forward response calculation of the perturbed model is performed to be able to compare the data residual of the forward computation with that of the original inversion model.

A question of this type arises, for example, in the common case where the interpreter wishes to find the minimum depth to a resistive or conductive basement, choosing a conductivity that qualifies what is meant by ‘resistive’ or ‘conductive’. The deeper the QDOI, the stronger conclusions can be made. It means that a basement halfspace of the tested conductivity cannot appear at a shallower depth. A more shallow QDOI only permits weaker conclusions to be drawn. Computationally, the QDOI is rather cheap: It only requires the calculation of one forward response for every test depth. The test depths are chosen as the layer boundaries of the multi-layer model; for example in the case of a 30-layer model, it means that, most often, 5–20 forward calculations need to be made.

Application of the QDOI measures on field data

Using the same flight line as the one in Figures 2 and 3, Figure 4 shows the two QDOIs relating to the conductivities of 1 S/m and 1 mS/m, that is for a conductive and a resistive halfspace. Comparing with Figure 2, the QDOIs are similar to the DOI estimates in large parts of the line. It is also seen that in the conductive parts of the line, the QDOI pertaining to a conductive halfspace is more shallow than the one for a resistive halfspace – as expected. In the more resistive parts of the line, the QDOI pertaining to the 1-mS/m halfspace lies higher than that of the 1-S/m halfspace.

DEPTH OF REQUIRED STRUCTURE

Defining the depth of required structure

In this section I will introduce a new measure of resolution, the depth of required structure (DORS). The DORS can be thought of as a general case of a QDOI. In the same way as the QDOI, it defines the minimum depth to a homogeneous halfspace, given that the model structure above that depth is

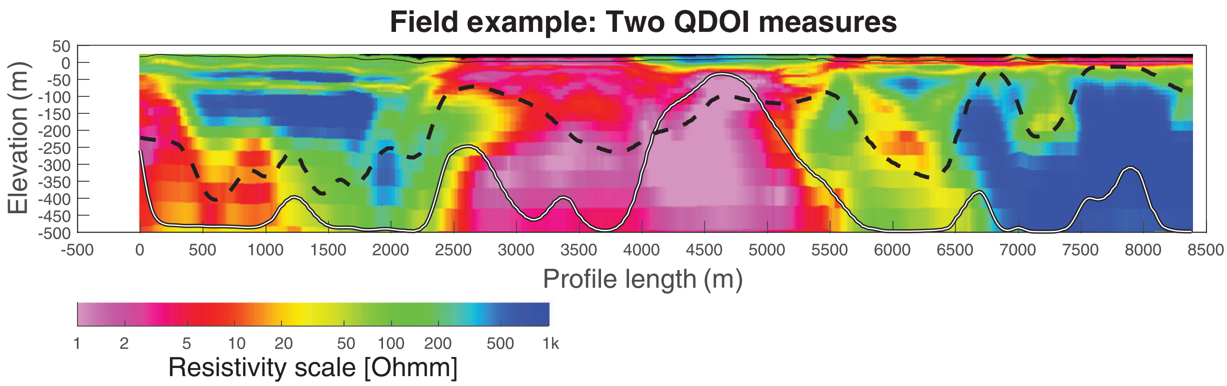

Figure 4 Model section with two QDOI definitions for the same model section as in Figure 2: One based on a conductivity of 1 S/m is indicated with a white curve with black edges, and one based on 1 mS/m is indicated with a dashed curve. The selection criterion is that the residual does not increase more than a factor of 1.2.

identical to the original inversion model and provided that the data residual of the perturbed model is smaller than a certain factor times (or absolute term added to) the one of the original inversion model. However, unlike the QDOI that is defined in relation to a specific halfspace conductivity, in the case of the DORS, the halfspace can have any conductivity; a conductivity that can – and does – change from one data location to the next. We thus have:

The DORS is the minimum depth to a homogeneous halfspace with any conductivity provided that the data residual must not increase more than a certain amount.

It is seen that the DORS indicates the depth below which no structure is needed to fit the data – structure understood as changes in conductivity with depth.

Understanding the DORS: What it is and what it is not

It is important to understand precisely what the DORS can be used for: what it is – and what it is not.

- The DORS indicates the depth above which you cannot dispense with structure without compromising the data fit. This means that the structures seen above the DORS can be considered as required, or quite well resolved. However, though the structure is required to fit the data, the usual principles of equivalence still apply, meaning that it is not the only model that will satisfy the conditions.
- In the case where the actual subsurface conductivity model is homogeneous below a relatively shallow depth, the DORS can become quite shallow because the model is actually homogeneous below that depth. In this case, the DORS does not indicate the bottom of the resolved structure which may be considerably deeper. It is important to keep this caveat in mind when using the DORS in the interpretation situation. However, the ambiguity can easily be resolved by inspecting the structure of the original multi-layer model from which the DORS was derived.
- The DORS does not indicate that there is no structure below the DORS. It says that the structures that might be present below the DORS are unnecessary to fit the data – not that they do not exist. The structures found below the DORS can be regarded as possibly being indicated, but so weakly that they can be dispensed with to fit the data. It is up to the interpreter to assign the proper significance to them.

Calculation of the DORS

Obviously, the DORS is computationally more expensive than QDOIs. Instead of one forward response for each of the test depths, several have to be tried to find the conductivity that minimizes the data residual for that particular test depth. If an exhaustive range of conductivities is considered between, for example 0.2 mS/m and 2 S/m, it takes 6–10 forward calculations for each test depth, so, in some cases, more than 100 forward responses need to be calculated for every model.

Application of the DORS measure to field data

In Figure 5, the DORS is plotted together with the DOI defined as the maximum centroid depth of the MRM kernels. By its very definition, the DORS is the most shallow of all the resolution measures presented in this paper. The figure shows that the DORS lies very close to the surface in the middle, very conductive part of the model section. It is an example of the second point in the list above: the shallow DORS does not mean that there is no resolution below the DORS depth; in this case, it signifies that the model is in fact close to

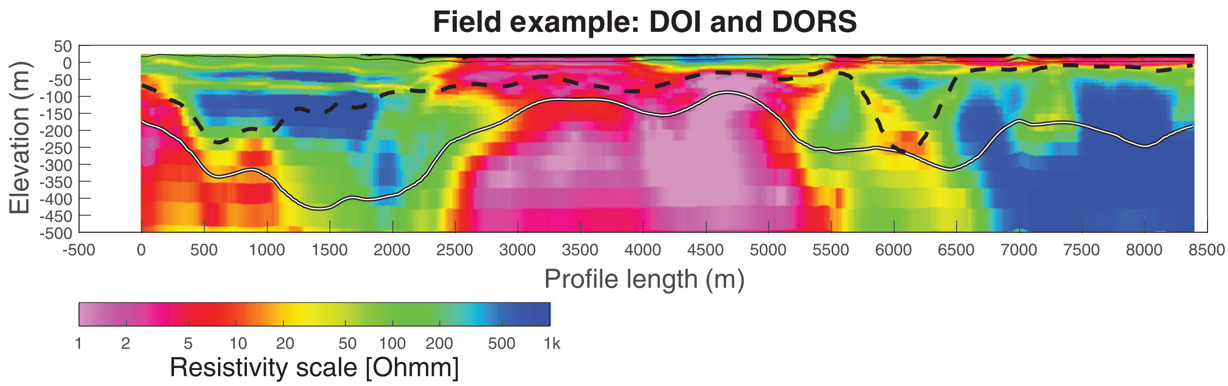

Figure 5 Model section with the DORS and DOI depths plotted on the section for the same model section as in Figure 2. The DOI defined as the maximum depth of the centroid is indicated as a white line with black rims. The DORS depth is the more shallow one of the two plotted as a dashed curve. The selection criterion is that the residual does not increase more than a factor of 1.2.

being a homogeneous halfspace below the DORS at this part of the section.

DISCUSSION

The resolution attributes presented here are intended to provide a meaningful guidance to the interpretation process. This requires of course that both inverter and interpreter understand the information content and significance of the measures. Correctly understood, all of the resolution measures presented in this paper can help interpreters evaluate the reliability and resolution of the models derived from inversion.

This process necessarily requires that a good dialogue can be established between inverter and interpreter in an iterative process that can lead to the best possible result. The attributes of the inversion models suggested here must be regarded as a help in this dialogue. They can be regarded as ‘words’ or ‘concepts’ in the ‘language’ between inverter and interpreter, and, as with any language, the proper use of the words must be learned through education and practicing by both inverter and interpreter. However, once learned, they can contribute to a better workflow and a more reliable interpretation.

CONCLUSIONS

Any interpretation depends critically on the interpreter being able to discern between the characteristics of the conductivity model provided by the inverter that can be trusted as more or less well resolved and the ones that are more dubious. To assist interpreters in their endeavours, several attributes of the inverted models have been suggested as indicators of resolution properties. The DOI is one such attribute that, ideally, indicates the depth below which model features cannot be trusted, thereby implicitly indicating the parts of the model where resolution is better. This paper has presented an investigation into some of the concepts that should be used to characterize a DOI measure and discussed published methods of defining the DOI, showing that many of them are suboptimal in relation to their purpose. Two novel DOI measures based on the MRM are suggested. They fulfil the criteria set-up in this paper: The defining concept is easy to understand and reasonable, it does not require an ad hoc limiting value to be defined by the user, it is a posterior measure taking data error and regularization into account and it is straightforward to calculate.

Further resolution measures can be derived from the MRM, and two of them have been presented here: the resolution width calculated as a L2 and a L1 measure. Model sections of the resolution width provide an insight into the variation of the resolution width with depth.

Two more concepts are introduced: a QDOI and the DORS. The first one answers a specific question: What is the minimum depth to a homogeneous halfspace with an interpreter-defined conductivity? It presents limits on geological features suggested by the inversion results. The second one is an ‘unqualified’ QDOI in the sense that the minimum depth to a homogeneous halfspace defined by the DORS does not relate to a certain conductivity; the halfspace can have any conductivity. Both the QDOI and the DORS need an ad hoc limiting value for the permitted increase in data residual.

The field examples given have been from transient AEM data, but the principles behind the definition of the resolution attributes have general validity, not only for EM methods, but for a wide variety of geophysical methods in general.

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DATA AVAILABILITY STATEMENT

The airborne transient data used in the figures of the field examples are open access, freely available and can be downloaded from Geoscience Australia's web site https://www.ga.gov.au/.

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APPENDIX

RESOLUTION MEASURES DERIVED FROM THE MODEL RESOLUTION MATRIX

This appendix addresses the computational issues involved in finding the resolution measures mentioned in the paper and presents analyses and tables that illustrate their nature. Several of the suggested resolution attributes depend on the moments of the rows (also referred to as resolution kernels) of the MRM that will be defined in the first section.

Moments weighted with the layer thicknesses

The nth moment of a distribution, $R(z)$, defined on the interval $z \in [0 : \infty]$ around the value z_0 is given as

$$\int_0^\infty R(z) \cdot (z - z_0)^n \, dz = 0. \tag{A1}$$

In the case of a discrete distribution of R, the zeroth, first and second moment around $z_0 = 0$ are thereby given as

$$M_0 = \int_0^\infty R(z) \cdot dz = \sum_{j=1}^{L} R_{ij} \cdot \int_{z_j}^{z_{j+1}} dz = \sum_{j=1}^{L} R_{ij} \cdot \left(z_{j+1} - z_j\right), \tag{A2}$$

$$M_1 = \int_0^\infty R(z) \cdot z \, dz = \sum_{j=1}^{L} R_{ij} \cdot \int_{z_j}^{z_{j+1}} z \, dz = \sum_{j=1}^{L} R_{ij} \cdot \frac{1}{2}\left(z_{j+1}^2 - z_j^2\right), \tag{A3}$$

$$M_2 = \int_0^\infty R(z) \cdot z^2 \, dz = \sum_{j=1}^{L} R_{ij} \cdot \int_{z_j}^{z_{j+1}} z^2 \, dz = \sum_{j=1}^{L} R_{ij} \cdot \frac{1}{3}\left(z_{j+1}^3 - z_j^3\right). \tag{A4}$$

In equations (A2)–(A4), both the continuous and the discrete versions are listed, the latter being the one to be used in our case where we consider L discrete values of the rows of the MRM, where L is the number of layers in the multi-layer model and z indicates depth. R_{ij} are the absolute values of the discrete values of the ith row and the j column, and z_j indicates the depth to the top of the jth layer. R_{ij} are chosen as the absolute value of the elements of the MRM since both positive and negative values must be regarded as equally contributing to the attributes.

Moments weighted with the layer numbers

The three moments mentioned above can also be derived based on a weighting with the layer numbers instead of the layer thicknesses, which is equivalent to using a layer thickness of 1 for all layers. In this case, the moments around $z_0 = 0$ will become

$$M_0 = \sum_{j=1}^{L} R_{ij}, \tag{A5}$$

$$M_1 = \sum_{j=1}^{L} R_{ij} \cdot \frac{1}{2}\left[(j+1)^2 - j^2\right], \tag{A6}$$

$$M_2 = \sum_{j=1}^{L} R_{ij} \cdot \frac{1}{3}\left[(j+1)^3 - j^3\right]. \tag{A7}$$

The question of whether to use a weighting with layer thicknesses or with layer numbers when calculating the moments is an important one and will be addressed below.

The DOI measure based on maximum values

The first of the DOI measures is defined as

$$\mathrm{DOI} = \max\{\max\{R_{ij}, i = 1, 2, \ldots, L\}, j = 1, 2, \ldots, L\}. \tag{A8}$$

Note that this definition does not depend on whether the moments are weighted with layer thicknesses or layer numbers.

The DOI measure based on centroid depth

The centroid *depth*, C, pertaining to a row of the MRM is defined as the depth value around which the first moment is zero:

$$\int_0^\infty R(z) \cdot (z - C)\,\mathrm{d}z = 0. \tag{A9}$$

We find:

$$\int_0^\infty R(z) \cdot (z - C)\,\mathrm{d}z = \int_0^\infty R(z) \cdot z\,\mathrm{d}z - C \cdot \int_0^\infty R(z)\,\mathrm{d}z \Rightarrow \tag{A10}$$

$$C = \int_0^\infty R(z) \cdot z\,\mathrm{d}z \Big/ \int_0^\infty R(z)\,\mathrm{d}z = M_1/M_0. \tag{A11}$$

The DOI measure is then defined as the maximum over the layers of the centroid depths:

$$\mathrm{DOI} = \max\{C_i, i = 1, 2, \ldots, L\}. \tag{A12}$$

Measures of the resolution width

A measure of the resolution width as a function of depth, that is the width of the resolution kernel for each layer, can be defined as the standard deviation, σ, defined by the second moment about the centroid depth. The second moment defines the variance of the resolution kernels and is a measure of the width that we are interested in. We then have

$$\begin{aligned}\sigma^2 &= \int_0^\infty R(z) \cdot (z - C)^2\,\mathrm{d}z \\ &= \int_0^\infty R(z) \cdot z^2\,\mathrm{d}z + C^2 \cdot \int_0^\infty R(z)\,\mathrm{d}z - 2C \cdot \int_0^\infty R(z) \cdot z\,\mathrm{d}z \\ &= M_2 + C^2 \cdot M_0 - 2C \cdot M_1,\end{aligned} \tag{A13}$$

where σ^2 is the variance.

This expression must be normalized with the integral of the distribution, M_0, otherwise it will depend on the amplitudes of the elements of the MRM, and we are only interested in the width. Substituting $C = M_1/M_0$, we find the normalized expression:

$$\sigma_n^2 = \sigma'^2/M_0 = M_2/M_0 + C^2 - 2C \cdot M_1/M_0 = M_2/M_0 - C^2, \tag{A14}$$

and thereby the normalized standard deviation:

$$\sigma_n = \sqrt{M_2/M_0 - C^2}. \tag{A15}$$

As a measure of the resolution width, we will use twice the standard deviation: $W_{L2} = 2 \cdot \sigma_n$.

Another resolution width measure shall be defined, namely the half-width of the kernels. This measure is defined as the width of the depth interval within which half of the integrated absolute kernel values are found. It is a sort of L1 standard deviation:

$$W_{L1} = H_{3/4} - H_{1/4}, \tag{A16}$$

where $H_{3/4}$ and $H_{1/4}$ are defined as

$$\int_0^{H_{1/4}} R(z)\,\mathrm{d}z = M_0/4 \quad \text{and} \quad \int_0^{H_{3/4}} R(z)\,\mathrm{d}z = 3 \cdot M_0/4. \tag{A17}$$

Table A1 MRM analysis for a resistive and a conductive model based on a weighting by layer numbers. DOI_1 is defined as the maximum layer number of the maximum value within each row of the MRM. DOI_2 is the DOI definition that refers to the centroid depth. The other columns are explained in the text

MRM analysis: Resistive model						MRM Analysis: Conductive model					
DOI_1 464.15		DOI_2 399.98	log(DOI_1/DOI_2) 0.15			DOI_1 464.15		DOI_2 399.98	log(DOI_1/DOI_2) 0.15		
Lay#	Depth	Resist	Centr	Stddev	$\frac{1}{2}$ Width	Lay#	Depth	Resist	Centr	Stddev	Width
1	0.00	120.6	5.56	6.05	5.00	1	0.00	24.6	4.58	5.73	4.78
2	2.00	98.2	5.03	5.30	3.35	2	2.00	13.7	4.35	5.00	3.65
3	4.04	72.5	4.32	4.30	2.84	3	4.04	7.2	4.75	4.26	3.00
4	6.17	58.9	4.70	4.44	2.87	4	6.17	3.6	4.51	3.25	1.72
5	8.44	61.9	6.55	5.04	4.06	5	8.44	2.1	4.82	2.67	1.43
6	10.90	95.9	7.80	4.88	5.86	6	10.90	2.1	5.82	2.86	1.55
7	13.60	190.5	8.00	4.68	6.19	7	13.60	2.0	6.83	2.66	1.49
8	16.60	381.6	8.45	4.77	6.74	8	16.60	3.4	7.61	3.03	2.00
9	19.97	527.8	8.95	5.01	7.13	9	19.97	3.4	8.34	3.01	2.10
10	23.78	562.0	9.64	5.25	7.39	10	23.78	2.3	9.35	2.80	1.54
11	28.12	476.7	10.59	5.33	7.14	11	28.12	3.7	10.41	3.83	1.85
12	33.09	314.3	11.71	5.10	5.79	12	33.09	10.1	11.57	4.53	3.31
13	38.78	153.0	12.78	4.51	3.38	13	38.78	17.4	12.78	4.63	4.03
14	45.34	58.0	13.82	3.36	1.77	14	45.34	15.5	13.53	4.29	3.96
15	52.89	34.7	14.30	3.72	1.83	15	52.89	7.8	14.63	3.65	2.38
16	61.63	58.6	15.75	3.20	2.15	16	61.63	3.6	15.24	3.38	1.60
17	71.72	52.7	15.69	3.84	2.36	17	71.72	4.2	16.78	3.67	2.14
18	83.41	38.4	17.15	3.38	1.68	18	83.41	8.3	17.83	4.30	3.26
19	96.94	31.7	17.73	3.50	1.36	19	96.94	10.4	18.77	4.23	3.39
20	112.62	75.3	18.95	3.36	1.60	20	112.62	6.8	19.47	3.86	2.57
21	130.80	311.2	19.25	3.92	2.88	21	130.80	3.8	20.03	3.92	1.94
22	151.88	970.7	20.14	4.83	4.36	22	151.88	4.7	19.79	5.19	2.39
23	176.32	1977.0	21.15	5.41	6.01	23	176.32	7.1	20.29	6.63	7.44
24	204.67	2640.2	22.17	5.64	7.22	24	204.67	8.2	21.62	7.10	7.02
25	237.55	2298.6	23.40	5.59	7.25	25	237.55	5.8	22.66	6.74	6.55
26	275.70	1355.9	24.87	5.06	5.26	26	275.70	2.0	23.32	5.83	4.84
27	319.96	610.7	26.30	3.93	2.57	27	319.96	0.3	23.59	5.00	4.72
28	371.30	273.1	26.83	3.39	1.69	28	371.30	0.2	23.33	5.55	5.30
29	430.88	193.4	27.14	3.43	1.52	29	430.88	0.7	22.51	6.36	6.52
30	500.00	187.7	27.16	4.83	1.97	30	500.00	2.2	21.26	7.13	7.35

Weighting with layer thickness or layer numbers?

Whether the MRM should be understood in terms of layer numbers or in terms of depth extent will have a considerable influence on the DOI defined by the centroid depth and on the resolution widths defined above. It is worth noting that, in and by itself, the MRM does not speak about layer thicknesses at all. The resolution kernel elements express the inverted parameter as a weighted sum of the true parameters for the given model discretization, meaning that the varying layer thicknesses are already inherently included in the value of the MRM. I am therefore most inclined to think of the MRM in terms of layers numbers. I have calculated values of the attributes defined above for both types of weighting, and my conclusion is that the weighting by layer numbers offers the best and most meaningful attributes. I will therefore recommend the use of moments based on the layer numbers in deriving the attributes.

DOIs and resolution width attributes of the resolution kernels: Two examples

Table A1 shows the resolution width measures defined above for all the layers of two 30-layer models: a fairly resistive and a fairly conductive one. In each of the two tables shown, the columns list the following: the layer number, the depth to the top of the layer, the layer resistivity, the centroid depth of the layer, the resolution width defined as two times the standard deviation and the half-width defined as the width of the central interval that contains half of the absolute values of the resolution kernel. In the header lines of the tables, both the DOI based on the maximum over all layers of the maximum of the resolution kernels (DOI_1) and the one based on the maximum over all layers of the centroid depths (DOI_2) are listed. The two DOI measures only differ by a limited amount: A relative difference of ~15%. When looking at the resolution width measures, that is the L2 and L1 half-widths of the resolution kernels, it must be kept in mind that they are in units of layer thicknesses.

It is seen from Table A1 that the L1 and L2 width measures are of the same magnitude, but there is no general rule as to which is the largest. However, the L1 resolution width has a greater dynamic range. (Also refer to Figure 1 showing the variation of the MRM row values.) Table A1 also shows that in the best resolved parts of the models: the middle depth range, especially where the resistivities are low, the L1 half-width is a number in the range 1 – 4, while the L2 half-width is a number in the range 2.5 – 4. This means that even for the best resolved parts of the model, the true parameters contributing most to the parameters of the inversion model come from the surrounding up to four layers, while in the more poorly poorer resolved parts it can be more than seven layers.

Comparing the resistive and the conductive models, it is seen that, as expected, the resolution widths are in fact smaller for conductive layers than for resistive layers; however, with the qualification that in a conductive model, the resolution widths start to increase in the deeper parts of the model that are shielded by the overlying conductive layers.