FAST HANKEL TRANSFORMS *

$\mathbf{B}\mathbf{Y}$

H. K. JOHANSEN ** and K. SØRENSEN ***

Abstract

JOHANSEN, H. K., and SØRENSEN, K., 1979, Fast Hankel Transforms, Geophysical Prospecting 27, 876-901.

Inspired by the linear filter method introduced by D. P. Ghosh in 1970 we have developed a general theory for numerical evaluation of integrals of the Hankel type:

$$g(r) = \int_{0}^{\infty} f(\lambda) \lambda J_{\nu}(\lambda r) d\lambda; \nu > - \mathbf{I}.$$

Replacing the usual sinc interpolating function by sinsh $(x) = a \cdot \sin(\pi x)/\sinh(a\pi x)$, where the smoothness parameter a is chosen to be "small", we obtain explicit series expansions for the sinsh-response or filter function H^* .

If the input function $f(\lambda \exp(i\omega))$ is known to be analytic in the region $0 < \lambda < \infty$, $|\omega| \leq \omega_0$ of the complex plane, we can show that the absolute error on the output function is less than $(K(\omega_0)/r) \cdot \exp(-\pi\omega_0/\Delta)$, Δ being the logarithmic sampling distance.

Due to the explicit expansions of H^* the tails of the infinite summation $\sum_{n=\infty}^{\infty} F(n\Delta)H^*$

 $((m-n)\Delta)$ can be handled analytically.

Since the only restriction on the order is $\nu > -\tau$, the Fourier transform is a special case of the theory, $\nu = \pm \tau/2$ giving the sine- and cosine transform, respectively. In theoretical model calculations the present method is considerably more efficient than the Fast Fourier Transform (FFT).

1. INTRODUCTION

This paper is a further development of the linear filter method introduced by Ghosh (1970, 1971, 1971a). The astoundingly good results of this simple and fast numerical transformation between apparent resistivity and resistivity transform curves inspired a number of contributions: A digital filter for computation of electromagnetic dipole sounding curves was published by Koefoed, Ghosh, and Polman (1972), and the method was extended to DC-dipole configurations by Das, Ghosh, and Biewinga (1974) and Das and Ghosh (1974). Improved sets of filter coefficients have been reported by Verma and Koefoed

^{*} Paper read at the thirty-ninth meeting of The European Association of Exploration Geophysicists, Zagreb, June 1977.

^{**} Dept. of Applied Geophysics, University of Luleå, S-951 87 Luleå, Sweden.

^{***} Laboratory of Geophysics, Finlandsgade 8, DK-8200 Aarhus N, Denmark.

(1973), Johansen (1975), Bichara and Lakshmanan (1976), Bernabini and Cardarelli (1978), and Koefoed and Dirks (1979). Koefoed (1972) analyzed theoretically the oscillatory behaviour of the filter function, and investigated error propagation in the direct interpretation using the Ghosh filter method (Koefoed 1976).

In all publications on the subject so far, the filter coefficients emerge as the ultimate result of extensive numerical computations. Besides obscuring the accuracy of the resulting digital filter this makes a subsequent analysis of the accuracy of the method rather difficult.

Searching for tools to construct an analytic solution we came across the book "Fourier Transform in the Complex Domain" by Paley and Wiener (1934) and found in it a complete mathematical theory for the Watson Transform (the author of "Theory of Bessel functions"), of which the transform with kernel exp (x) J, $(\exp (x))$ used by Koefoed, Ghosh, and Polman (1972) is a special case. We can recommend this book as a general reference to the following paragraph.

2. Reformulation of the Hankel Transform

We consider the Hankel integral

$$g(r) = \int_{0}^{\infty} f(\lambda) \lambda J_{\nu}(\lambda r) d\lambda, \qquad (1)$$

where J_{ν} is the Bessel function of the first kind, for real values of $\nu > -1$.

If we perform the substitutions

$$\lambda = \exp((-u); \quad r = \exp(v); \quad u, v \text{ in } (-\infty, \infty)$$
(2)

and define new functions F and G on $(-\infty, \infty)$ by

$$F(u) = \exp((-u)f(\exp((-u))); \qquad G(v) = \exp(v)g(\exp(v)), \qquad (3)$$

equation (1) may be written

$$G(v) = \int_{-\infty}^{\infty} F(u)H_{\nu}(v-u)\mathrm{d}u; G = F^*H_{\nu}$$
(4)

i.e., G is the convolution of F and the *kernel* of the transform

$$H_{\nu}(v) = \exp(v) \mathbf{J}_{\nu} (\exp(v)). \tag{5}$$

From computed values of G we can obviously retrieve values of g by a simple algebraic operation.

In order to see why the exponential function factors in the definition of F, G, and H are introduced in this particular way we must take the Fourier transform of eq. (4).

Using the convolution theorem (Bracewell 1965 p. 108) we get

$$\hat{G}(s) = \hat{F}(s) \cdot \hat{H}_{\nu}(s) \tag{6}$$

where

$$\hat{H}_{\nu}(s) = \int_{-\infty}^{\infty} H_{\nu}(v) \exp\left(-i2\pi v s\right) dv \tag{7}$$

and analogously for F and G.

Introducing the explicit expression (5) for H_{ν} and substituting back $v = \ln(t)$; $t = \exp(v)$ in (7), we get

$$\hat{H}_{\nu}(s) = \int_{0}^{\infty} J_{\nu}(t) t^{-i2\pi\delta} dt = 2^{-i2\pi\delta} \frac{\Gamma((\nu + \mathbf{I})/2 - i\pi s)}{\Gamma((\nu + \mathbf{I})/2 + i\pi s)}$$
(8)

where the rightmost equality is taken from Abramowitz and Stegun (1965) eq. 11.4.16. Conditions for convergence in (8) are

$$\Re e(-i2\pi s) < \frac{1}{2}; \qquad \Re e(\nu - i2\pi s) > -1 \tag{9}$$

which for real s and v are obviously fulfilled for all v > -1. Since the gamma function obeys the Schwarz reflection principle (AS 6.1.23)

$$\Gamma(\bar{z}) = \Gamma(z) \tag{10}$$

where the bar means complex conjugation, we see that \hat{H}_{ν} has the important property

$$|\hat{H}_{\nu}(s)| \equiv 1; \quad \nu > -1; \quad \mathfrak{Fm}(s) = 0.$$
 (11)

Taking the modulus of (6) we get

$$|\hat{G}(s)| = |\hat{F}(s)| |\hat{H}_{\nu}(s)| \equiv |\hat{F}(s)|,$$
 (12)

i.e., the input function F and output function G have identical spectra. Hence "noise" in F is neither amplified nor diminished by the convolution with H.

Since $\hat{H}_{\nu}(s)$ is evidently never equal to zero, we obtain algebraically from (6)

$$\widehat{F}(s) = \widehat{G}(s)/\widehat{H}_{\nu}(s) = \widehat{G}(s) \cdot \widehat{H}_{\nu}(s), \qquad (13)$$

where the second equality follows from the property (II). The inverse transform which carries G into F is now obtained by a further Fourier transformation of (I3):

$$F(v) = \int_{-\infty}^{\infty} G(u) \overline{H_{\nu}(-(v-u))} du; F = G^* \overline{H}_{\nu}^{-}$$
(14)

where $H_{\nu}(v) = H_{\nu}(-v)$ is just the original transform kernel "read backwards" or—more precisely—reflected in the origin.

878

The reason for the particular choice (5) of convolution kernel may now be justified:

I) The expression (8) is valid for all v > -I, which is the range of v encountered in most theorems involving Bessel functions. This means that the results are applicable not only for particular choice v = 0, I which gives the Hankel transforms of integer order, but also for $v = \pm I/2$, which correspond to the sine- and cosine transform, respectively, so that the theory contains the Fourier transform as well.

2) Since $|\hat{H}_{\nu}(s)| \equiv I$, the existence of the inverse transform follows immediately, and the symmetry of the Fourier- and Hankel theorems is preserved. The fact that there is no error amplification is particularly important and leads to substantial simplifications, as we shall see below.

3. FORMULATION OF THE NUMERICAL PROBLEM

The infinite sampling density, infinite representation accuracy and infinite integration range required for the exact evaluation of (4) can not be realized in a computer. A computer calculation can yield only an approximation G^* to G. Our numerical problem may thus be stated as follows: Construct an efficient algorithm which for a given ε (larger than the representation error in the computer) will calculate values of $G^*(v)$ so that $|G(v) - G^*(v)| < \varepsilon$ for all v in **I**. **I** being a *specified* subset of $(-\infty, \infty)$ which we find interesting.

We shall first be interested in constructing a G^* at equidistant points $v = m\Delta$, $m = 0, \pm 1, \pm 2, \ldots$, and secondly we shall consider a G^{**} interpolating between these values.

Let us construct an approximation F^* to F from equidistantly sampled values $F(n\Delta)$ by the interpolation scheme

$$F^*(u) = \sum_{-\infty}^{\infty} F(n\Delta) P(u/\Delta - n).$$
 (15)

At this point we do not want to commit ourselves to a particular interpolating function P, but one may think of P as "something like the sinc function", i.e. P(n) = 0 for the integer n different from zero and P(0) = I.

We define G^* by replacing F by F^* in (4)

$$G^*(v) \equiv \int_{-\infty}^{\infty} F^*(u) H_{\nu}(v-u) \mathrm{d}u = \sum_{-\infty}^{\infty} F(n\Delta) H_{\nu}^*(v-n\Delta), \qquad (16)$$

where

$$H_{\nu}^{*}(v) = \int_{-\infty}^{\infty} P(u/\Delta)H_{\nu}(v-u)\mathrm{d}u \tag{17}$$

is the *P*-response of the H_{ν} -transform.

In principle G^* may be evaluated for arbitrary v by (16), provided H is known in the corresponding set of points. In practice the overall computational work may be reduced considerably by evaluating only the equidistant values

$$G^*(m\Delta) = \sum_{-\infty}^{\infty} F(n\Delta) \cdot H^*_{\nu}((m-n)\Delta)$$
(18)

and later interpolate between them by an approximation G^{**} to G^{*} , for example of the same form as eq. (15):

$$G^{**}(v) = \sum_{-\infty}^{\infty} G^{*}(m\Delta) P(v/\Delta - m)$$
(19)

(the interpolating function used here may differ from the one used above). The essential problem is to compute values of H_{ν}^* .

4. An Analytical Expression for $H^*(v)$

The trick is to express $H_v^*(v)$ as a Fourier integral and evaluate it as a contour integral in the complex plane. Since (17) is a convolution, we get immediately

$$H_{\nu}^{*}(v) = \int_{-\infty}^{\infty} \Delta \hat{P}(\Delta s) \hat{H}_{\nu}(s) \exp(i2\pi v s) ds, \qquad (20)$$

where $\hat{P}(s)$ is the Fourier-transform of P(u). The properties of $\hat{H}_{\nu}(\zeta)$ for complex values of the argument $\zeta = s + i\sigma$ are investigated in appendix A. It is analytic in the entire complex plane except in the points $\zeta_n = (-(\nu + \mathbf{I})/(2 - n)i/\pi)$ on the negative imaginary axis where it has simple poles with residues

$$R_n = (i/\pi) (-I)^n 2^{-(2n+\nu+1)} / (\Gamma(n+I)\Gamma(\nu+n+I)); n = 0, I, 2, \dots$$
(21)

In the lower halfplane $|\hat{H}_{\nu}(\zeta)|$ goes to zero as $\sigma \to -\infty$ faster than any exponential function except at the poles, of course. This suggests that we should choose the integration contour C^- shown in fig. I. Applying Cauchy's theorem to the integral along C^- yields

$$\int_{C^{-}} \Delta \hat{P}(\Delta \zeta) \hat{H}_{\nu}(\zeta) \exp\left(i2\pi v\zeta\right) d\zeta = 2\pi i \sum_{C^{-}} \operatorname{Res}\left\{\Delta \hat{P}(\Delta \zeta) \hat{H}_{\nu}(\zeta) \exp\left(i2\pi v\zeta\right)\right\}$$
(22)

where the summation takes into account the residues at poles enclosed by C^- . Let for a moment $\Delta \hat{P}(\Delta \zeta) \equiv \mathbf{I}$, which is analytic everywhere. Then C^- may be extended to infinity in the lower halfplane, of the four contributions from C^- only the part along the real axis survives, and we get

$$\int_{-\infty}^{\infty} \Delta \hat{P}(\Delta s) \hat{H}_{\nu}(s) \exp (i2\pi v s) ds = -2\pi i \sum_{n=0}^{\infty} \{\Delta \hat{P}(\Delta \zeta_n) \cdot R_n \cdot \exp (v(2n+\nu+1))\}$$
(23)



Fig. 1. Distribution of poles of \hat{H}_{ν} and \hat{P} in the complex $\zeta = s + i\sigma$ plane. C^{-} and C^{+} are integration paths.

Inserting R_n from (21) and remembering $\Delta \hat{P}(\Delta \zeta) \equiv 1$ we recognize the right hand side of (23) as the series expansion of exp $(v) J_{\nu} (\exp(v)) = H_{\nu}(v)$.

The integration method we use has thus succeeded in giving us the result that the Fourier transform of \hat{H}_{ν} is H_{ν} itself, which is not too surprising. What makes the method interesting is that it gives an explicit series expansion for H_{ν}^{*} even when \hat{P} is not so simple.

If we choose P to be the sinc function, then $\hat{P}(\zeta) = \Pi(\zeta)$, equal to I if $|s| < \frac{1}{2}$, $\frac{1}{2}$ if $|s| = \frac{1}{2}$, and zero otherwise (Bracewell 1965). Hence $\Delta \Pi(\Delta \zeta)$ is only analytic in the strip $|s| < s_c$, where $s_c = 1/2\Delta$ is the cutoff- or Nyquist frequency, so that the integration path C^- must be confined to this strip. We end up with the right hand side of (23) *plus* the integrals along the two vertical parts with $s = \pm s_c$ of C^- , and these integrals have to be evaluated by numerical integration. This is essentially the idea of Baranov (1976).

In order to avoid numerical integrations and make use of (22) instead, we must make it possible for the contour C^- to pass the barriers at $s = \pm s_c$ and go to infinity. Hence we *must* replace Π by a function with "smooth" corners, or more precisely—by a function \hat{P} which is analytic all the way to infinity, except possibly in some isolated points not on the real axis.

Here we choose \hat{P} to be the product of two Fermi distributions of complex argument:

$$\hat{P}(s+i\sigma) = d[\mathbf{I} + \exp\left((s - \frac{1}{2} + i\sigma)2\pi/a\right)]^{-1} [\mathbf{I} + \exp\left((-s + \frac{1}{2} - i\sigma)2\pi/a\right)]^{-1}$$
(24)

where d is a normalization constant.

The behaviour of $|\hat{P}|$ on the real axis is shown in fig. 2 for various values of the "smoothness" parameter a. We see that \hat{P} becomes boxcar-like when a



Fig. 2. Dependence of $\hat{P}(s)$ on the smoothness parameter a. $\hat{P}(s)$ becomes boxcar-like when a goes to zero.

is small, so we will expect the corresponding interpolation function P to resemble the sinc-function.

d is determined from the demand

$$P(\mathbf{o}) = \int_{-\infty}^{\infty} \hat{P}(s) \mathrm{d}s = \mathbf{I}$$
 (25)

to be $d = 1 - \exp(-2\pi/a)$. Invoking this constant in (24) we find the inverse Fourier transform of $\hat{P}(s)$ to be

$$P(u) = a \cdot \frac{\sin(\pi u)}{\sinh(\pi a u)} \equiv \sinh(u), \qquad (26)$$

which is evidently reminiscent of the sinc-function. Hence "sinsh" would be a suitable mnemonic for it. An investigation of its behaviour is given in appendix B. We shall only mention one interesting property here: Applying the convolution theorem "read backwards" to (26), we find

$$\widehat{P}(s) = \frac{1}{2} \tanh\left((s + \frac{1}{2})\pi/a\right) - \frac{1}{2} \tanh\left((s - \frac{1}{2})\pi/a\right), \tag{27}$$

which of course may be verified directly from (24) and (25). This transscription will be useful later on.

Properties of $\Delta \hat{P}(\Delta \zeta)$ in the complex $\zeta = s + i\sigma$ plane are derived in appendix B. In the strip between the two vertical lines $s = \pm s_c$ it is almost equal to one, outside the strip it decays exponentially to zero, and on the two lines it has simple poles in the equidistant points

$$\zeta_k^{\pm} = \pm s_c - i(k + 1/2)2as_c; k \text{ integer}$$
(28)

with the *k*-independent residues

$$R_k^{\pm} = \pm 2as_c/2\pi. \tag{29}$$

Except for these poles $\Delta \hat{P}(\Delta \zeta)$ is analytic in the whole ζ -plane, so that we may apply (22). When we let the vertical parts of C^- go to plus and minus infinity, their contributions will go to zero even more rapidly than in the case considered above because of the exponential decay of \hat{P} in those directions. If we let the lower horizontal part of C^- pass exactly midway between the poles (28), \hat{P} is purely real, positive and less than one, so if we let the path cangaroo over the poles in this manner on its way to infinity in the lower halfplane, its contribution goes to zero by the *same* argument which lead to the Fourier inversion result above. Hence we obtain for $H^*_v(v)$ the expression at the right hand side of (23) *plus* the residues at the *extra* poles (28) with $k \ge 0$, which we have introduced in the lower halfplane by our more complicated choice of \hat{P} .

Let us consider only values of the smoothness parameter a in \hat{P} for which

$$\Delta/a = M\pi, \tag{30}$$

where M is a fixed positive integer. It is possible to carry out the calculations without this restriction, but it does not harm the results, and it leads to important simplifications both in the theory and in the practical calculations.

With this restriction on *a* it follows immediately that the numbers $\Delta \hat{P}(\Delta \zeta_n)$, where ζ_n are the poles (A2) of $\hat{H}_{\nu}(\zeta)$, are independent of *n*. Hence the sum of the residues on the imaginary axis becomes simply (see the right hand side of (23)) equal to a constant times $\hat{H}_{\nu}(v)$, the unmodulated convolution kernel. Because of the properties $\hat{H}(-s+i\sigma) = \overline{\hat{H}(s+i\sigma)}$ and $\hat{P}(-s+i\sigma) = \overline{\hat{P}(s+i\sigma)}$ (see app. A and B) we may combine the two remaining sums into one, so that we finally obtain

$$H_{\nu}^{*}(v) = c \cdot H_{\nu}(v) + \operatorname{Im}\{A_{\nu}(v) \cdot \exp(i2\pi s_{c}v)\}, \qquad (31)$$

$$c = (\mathbf{I} - \exp(-4M\pi^2 s_c)) \cdot \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} (-\mathbf{I})^{l+m+(\nu+1)M(l-m)} \exp(-2M\pi^2 s_c(l+m))$$
(31a)

where the oscillating exponential factor, which is common to all terms in the summation, has been isolated from the amplitude factor

$$A_{\nu}(v) = -M\pi \sum_{k=0}^{\infty} \hat{H}_{\nu}(s_{c} - i(k + 1/2)/M\pi) \exp(-v(k + 1/2)/M).$$
(32)

Equation (31) is an analytic expression for $H_v^*(v)$, the two series for $H_v(v)$ and $A_v(v)$ being absolutely convergent for all v in $(-\infty, \infty)$. Notice the oscillatory nature of the second term in (31), the "modulation" term. This phenomenon was studied by Koefoed (1972) by another mathematical technique. Here we have succeeded in establishing an explicit series expansion of the amplitude of the oscillation, which moreover is very rapidly convergent for negative v. We shall investigate the nature of the residues along the vertical line $s = s_c$ a little closer. The poles (28) lie at equidistant points $2as_c$ apart from each other. The residues of the modulation function $\Delta \hat{P}(\Delta \zeta)$ are numerically equal to this distance divided by 2π . When we form the right hand side of Cauchy's theorem, which is $2\pi i$ times the sum of residues, the 2π factors cancel each other, and we are left with the sum of equidistantly sampled values of the function $\hat{H}_{\nu}(\zeta) \exp(i2\pi v \zeta)$ times the sampling distance. This is, however, precisely the sum one would form in a numerical integration procedure with equidistant sampling at the midpoints of the subintervals. In the limit where the smoothness parameter a in \hat{P} goes to zero, $\hat{P}(s)$ becomes equal to $\Pi(s)$ on the real axis, and the vertical summations become integrations, all in nice harmony with what was said before eq. (24). Conversely, if we had adopted the sinc-function as interpolating function, and had chosen to perform the numerical integrations as described here, we would *actually* have used the present sinsh-function theory, but perhaps without realizing it.

5. Asymptotic Expansion of $H^*_{\nu}(v)$

The absolute convergence of (31) for all real values of v is a nice mathematical property, but unfortunately this does not at all secure a nice behaviour in the computer. When v becomes positive and large, the absolute value of the kth term in the series (32) increases rapidly for a number of terms, until eventually the faster-than-any-exponential drop of $|\hat{H}_v|$ becomes dominating, and the series converges. In a computer the result of a summation can not be expected to be more accurate than the representation error of the numerically largest term. Actually, the value of the modulation term for large v computed using (32) would be the sum of a huge number of large terms which cancel each other more or less. In order to avoid trouble the absolute value of the terms should be (essentially) decreasing for increasing k. Using (A7) it can be shown that this is the case when $v \leq \ln (2\pi s_c)$.

The well-known remedy against this problem for the Bessel function—which is the important ingredience in H_{ν} , the first term in (31)—is to make an asymptotic expansion. We shall do this for H_{ν}^* as well.

When things go wrong in the lower halfplane we are left with no alternative but the upper halfplane. Hence we consider the integration path C^+ in fig. 1. The relations

$$\exp(i2\pi v(s+i\sigma)) = \overline{[\exp(i2\pi v(s-i\sigma))]^{-1}} \text{ and } H_{\nu}(s+i\sigma) = [\widehat{H}_{\nu}(s-i\sigma)]^{-1}$$

(see (A5)) connect the values of the integrand in the upper halfplane with those taken at the complex conjugated points in the lower halfplane. The modulus of the exponential factor is exp $(-2\pi v\sigma)$, which for positive v is decaying in the upper halfplane, and more strongly so when v is large. For $\sigma \rightarrow +\infty \mid H_v \mid$

884

increases more rapidly than any exponential, so that the modulus of the integrand blows up, but it has a minimum for a certain σ_m , which depends on the value of v.

Hence we choose the horizontal return path of C^+ to be $\zeta = s + i\sigma_m$. Due to the exponential decay of $|\hat{P}|$ outside the interval between $-s_c$ and $+s_c$ we may let the vertical parts of C^+ go to infinity, where they contribute nothing to the integral. The "error integral" from the horizontal return path goes to zero when v goes to infinity. Hence the sum of the residues in the upper halfplane forms an asymptotic series for H^*_{ν} :

$$H^*_{\nu}(v) \sim \operatorname{Im}\{B_{\nu}(v) \exp\left(i2\pi s_c v\right)\}$$
(33)

where the amplitude factor B_{ν} is given by

$$B_{\nu}(v) \sim M\pi \sum_{k} \hat{H}_{\nu}(s_{c} + i(k + 1/2)/M\pi) \exp(-v(k + 1/2)/M), \quad (34)$$

the summation in k extending over those poles which are enclosed by C^+ .

It can be shown—using (A8)—that one may choose $\sigma_m = \exp(v)/2\pi$ which gives an upper limit to the error integral:

$$| \text{ error integral} | \leq 2 \cdot \exp(v - \exp(v)) \cdot \exp(\pi^2 s_c) / \pi^2 s_c.$$
(35)

For any $\varepsilon > 0$ one can determine v^+ so large that the right hand side of (35) is smaller than ε , and hence the expressions (33) and (34) give $H^*_{\nu}(v)$ with error less than ε for all $v \ge v^+$.

6. Calculation of $H^*_{\nu}(v)$ in the Interval $v^- < v < v^+$

In this interval neither of the two methods presented above are applicable, so we have to introduce a trick based on the properties of the function

$$\widehat{Q}(\zeta) = (\mathbf{I} + \exp\left(-\frac{i2\pi}{b} \cdot (\zeta - i\sigma_c)\right))^{-1}; \quad \zeta = s + i\sigma$$
(36)

which is examined in appendix C. It has the same form as the two factors forming $\hat{P}(\zeta)$, only rotated ninety degrees, so that the poles lie equidistantly on the horizontal line $\sigma = \sigma_c$:

$$\zeta_n = (n + 1/2)b + i\sigma_c; \quad n = \dots -2, -1, 0, 1, 2 \dots$$
 (37)

with equal residues R_n

$$2\pi i R_n = b. \tag{38}$$

The trick is to consider the integral

$$H_{\nu}^{**}(v) = \int_{-\infty} \hat{Q}(s) \Delta \hat{P}(\Delta s) \hat{H}_{\nu}(s) \exp(i2\pi v s) ds$$
(39)

instead of (20). For a fixed $\sigma_c > 0$ we can choose b so small that $|\hat{Q}(s) - \mathbf{I}| < 10^{-16}$ on the real axis. Hence the difference between the old integrand in (20) and the new integrand in (39) is less than the representation error on most compu-Geophysical Prospecting 27 58 ters. It is easy to show that this holds for the difference between H_{ν}^* and H_{ν}^{**} as well, so we may replace H_{ν}^* by H_{ν}^{**} .

We evaluate H_{ν}^{**} by integration along the path C^+ shown in fig. 3. The contributions from the vertical parts of C^+ go to zero as before, and the error integral along the horizontal return path contains an extra exponential decay factor exp $(-2\pi/b(\sigma_m - \sigma_c))$ which—with the three parameters b, σ_c and σ_m properly chosen—make the error integral less than a given ε . Hence H_{ν}^{**} is obtained as the "asymptotic series" of residues from the vertical poles *plus* the sum of residues introduced at the horizontal poles of \hat{Q} .



Fig. 3. Positioning of the "horizontal" poles from \hat{P} relative to the "vertical" poles from \hat{Q} .

It is interesting to interpret this extra summation as a numerical integration, just as we did in section 4. If we let σ_c go to zero the poles approach the real axis. In order to still have $|\hat{Q}| \sim I$ on the real axis we must let b go to zero as well. Hence the sampling scheme becomes infinitely dense so that the sum of residues equals the integral on the real axis. In practice we want to reduce the numbers of sampling points, so we must choose σ_c as large as possible in order to permit a large sampling distance b while still preserving $|\hat{Q}| \sim I$ on the real axis with sufficient accuracy. The present application of this integration technique gives ten correct decimal digits in H_v^{**} at the expense of about twenty "horizontal" sampling points in the complex plane.

7. DEPENDENCE OF THE sinsh-Response $H_{\nu}^{*}(v)$ on s_{c} and M

In order to avoid unnecessary complications we shall consider only v = -1/2, for which we have $H_{-1/2}(v) = (2/\pi)^{1/2} \exp(v/2) \cos(\exp(v))$. This is a permissible simplification because the $H_{\nu}(v)$ for different ν —apart from a phase term which depends only on ν in the argument of the cosinus—show the same asymptotic behaviour for $v \to \infty$. Due to the exponential argument of the







Fig. 5. Graph of $H^*_{-1/2}$ for M = 2 and $s_c = 2\pi$.



Fig. 6. Three dimensional view of $H^*_{-1/2}$ with M = 2 and v and s_c as variables.



Fig. 7. Dependence of $H_{-1/2}^*$ on M. s_c is equal to π . The smoothest curve corresponds to M = I while the more oscillating curve corresponds to M = Io.

cosine, arbitrarily high frequencies become represented in $H_{\nu}(v)$ (and due to the exponential envelope with modulus one) as $v \to \infty$, so it is this part of the integral kernel which lets through high frequencies in the input function F. Fig. 4 shows the sinsh-response function $H_{\nu}^{*}(v)$ for the "pseudo cutofffrequency" s_c equal to π . We notice that about v = 2 the asymptotic expression (33) takes over, thus admitting practically no frequencies above s_c to pass. For negative values of v, however, the modulation term is merely a perturbation (with characteristic frequency s_c) to the undisturbed integral kernel. In fig. 5 s_c is twice as big. The relative magnitude of the modulation term for negative v has diminished, while the original integral kernel now survives up to about v = 3, before (33) takes over. The gradual recovery of the high frequency part of $H_{-1/2}(v)$ with increasing s_c is perhaps illustrated best by the three dimensional view in fig. 6. The dependence on the "sharpness number" M is less complicated, as depicted in fig. 7, where two curves corresponding to M = 10, M = 1 are drawn. For large M the magnitude of the modulation term becomes larger and dies out more slowly for $|v| \rightarrow \infty$.

8. AN UPPER BOUND ON THE SAMPLING ERROR

We can obtain an upper bound for the error $|G(v) - G^*(v)|$ in the following way: Replace $G - G^*$ by the inverse fourier transform of $\hat{G} - \hat{G}^*$, move the modulus sign inside the integral and omit the modulus of the exponential, since it is equal to one. This gives the inequality:

$$|G(v) - G^*(v)| \leq \int_{-\infty}^{\infty} |\widehat{G}(s) - \widehat{G}^*(s)| \, \mathrm{d}s \tag{40}$$

where the right hand side is independent of v. Since $G^* = F_x^* H_v$ (eq. 16), we have $\hat{G}^* = \hat{F}^* \cdot \hat{H}_v$, which together with (6) gives

$$|\hat{G} - \hat{G}^*| = |\hat{F} - \hat{F}^*|$$
 (41)

because $|\hat{H}_{\nu}| \equiv I$.

In order to express \hat{F}^* in terms of \hat{F} and \hat{P} we consider an auxiliary generalized function \mathbf{F} defined by means of the *shah* symbol $\underline{\text{TTT}}$, which is a series of equispaced Dirac delta functions.

$$\mathbf{F}(u) = F(u) \cdot (\mathbf{I}/\Delta) \prod (u/\Delta) \tag{42}$$

where (Bracewell 1965 p. 77)

$$\underline{\Box} \underline{\Box} (u) = \sum_{-\infty}^{\infty} \delta(u-n).$$
(43)

We observe that

$$F^*(u) = \mathbf{F}(u) * P(u/\Delta). \tag{44}$$

Using the convolution theorem we get:

$$F^*(u) = \widehat{\mathbf{F}}(u) \cdot \Delta \widehat{P}(\Delta s). \tag{45}$$

The fourier transform of $(I/\Delta) \coprod (u/\Delta)$ is $\coprod (\Delta s)$ (Bracewell p. 79), so when we apply the convolution theorem to (42) we obtain

$$\widehat{\mathbf{F}}(s) = \widehat{\mathbf{F}}(s) \cdot \prod (\Delta s) = (\mathbf{I}/\Delta) \sum_{-\infty}^{\infty} \widehat{F}(s - n/\Delta)$$
(46)

and hence finally

$$\widehat{F}^*(s) = \widehat{P}(\Delta s) \sum_{\infty}^{\infty} \widehat{F}(s) - n/\Delta).$$
(47)

Inserting (41) and (47) in (40) and splitting the term with n equal to zero from the summation we obtain

$$|G(v) - G^*(v)| \leq I_1 + I_2,$$
 (48)

where

$$I_{1} = \int_{-\infty}^{\infty} |\hat{F}(s)| | \mathbf{I} - \hat{P}(\Delta s) | \mathrm{d}s$$
(49)

and

$$I_{2} = \int_{-\infty}^{\infty} |\hat{P}(\Delta s)| \cdot \sum_{n \neq 0} |\hat{F}(s - n/\Delta)| \, \mathrm{d}s.$$
 (50)

Since $\hat{P}(\Delta s)$ and $\mathbf{I} - \hat{P}(\Delta s)$ are both real and positive in this case the modulus signs on them may be left out.

The integral I_2 can be simplified considerably: We exchange the order of integration and summation in (50), change the integration variable in each of the integrals and exchange the summation and integration once more. We obtain

$$I_2 = \int_{-\infty}^{\infty} \{ \sum_{n \neq 0} \hat{P}(\Delta s + n) \} \cdot |\hat{F}(s)| \, \mathrm{d}s.$$
(51)

From (B 13) we see that the function in brackets is nothing but $I - \hat{P}(\Delta s)$, so I_2 —which is due to aliasing—is actually equal to I_1 , which comes from truncation of the spectrum. The validity of the above operations is ensured by absolute convergence of the integral (50). For functions F meeting this demand we have thus achieved a global error limit:

$$|G(v) - G^*(v)| \leq 2 \int_{-\infty}^{\infty} |\hat{F}(s)| (\mathbf{I} - \hat{P}(\Delta s)) ds$$
(52)

9. Analyticity of f and the Spectrum of F

We shall prove the following spectrum majorization theorem:

If $f(z) = f(\lambda \exp(i\omega))$ is analytic in the region $0 < \lambda < \infty$, $|\omega| \le \omega_0$ of the complex plane, if f(z) is $O(|z|^{-1+\varepsilon})$, $\varepsilon > 0$ for $|z| \to 0$ and is $O(|z|^{-2-\delta})$, $\delta > 0$ for $|z| \to \infty$ in this region, then

$$|\hat{F}(s)| \leq K(\omega_0) \exp\left(-2\pi\omega_0 |s|\right)$$
(53)

for all s in $(-\infty, \infty)$, where

$$K(\omega_0) = \max \{ \int_{0}^{\infty} |f(\lambda \exp(\pm i\omega_0))| d\lambda \}$$
(54)

This theorem is a direct consequence of the exponential substitution (2) and Cauchy's integral theorem. We consider the rectangular integration path ABCD in fig. 8, which is mapped into the angular circle section *abcd* by (2). If f is



Fig. 8. Correspondence between the domains for f and F. The logarithmic substitution maps the integration path abcd into ABCD (and vice versa for the exponential substitution).

analytic inside *abcd*, then F will be analytic inside *ABCD*. The integrals of |f| along *bc* and *da* vanish when we let *b* and *c* go to the origin and *a* and *d* go to $+\infty$ and $+\infty \cdot \exp(-i\omega_0)$, respectively. Hence the integral of |F| along *BC* and *DA* also vanish when these paths are at plus and minus infinity, respectively. Applying Cauchy's theorem to the Fourier integral of *F* we thus get

$$\widehat{F}(s) = \int_{-\infty}^{\infty} F(u) \exp\left(-i2\pi us\right) du = \int_{-\infty}^{\infty+i\omega_0} F(w) \exp\left(-i2\pi ws\right) dw$$
$$= \exp\left(+2\pi\omega_0 s\right) \int_{-\infty}^{\infty} F(u+i\omega_0) \exp\left(-i2\pi us\right) du$$
(55)

so that

$$|\widehat{F}(s)| \leq \exp\left(+2\pi\omega_0 s\right) \int_{-\infty}^{\infty} |F(u+i\omega_0)| \,\mathrm{d}u.$$
(56)

The s-dependence of the right hand side is entirely in the exponential function, which decays for decreasing s with decay constant $2\pi\omega_0$ proportional to the opening angle ω_0 of the region of analyticity. The inequality is valid for all s in $(-\infty, \infty)$, but it is only useful for negative s. In order to obtain a useful inequality for positive s as well, we choose the dashed integration contour in fig. 8, which only changes ω_0 into $-\omega_0$ in (54). The two inequalities can be combined into one:

$$|\hat{F}(s)| \leq K(\omega_0) \exp\left(-2\pi\omega_0 \mid s \mid\right)$$
(57)

where $\omega_0 > 0$ and

$$K(\omega_0) = \max \{ \int_{-\infty}^{\infty} |F(u \pm i\omega_0)| du \}.$$
 (58)

Inserting F from (3) and changing variable according to (2) we observe that the present expression for $K(\omega_0)$ is identical with the one postulated in the theorem, and this completes the proof. Notice that f need not be analytic at the origin, we only demand that f(z) be $O(|z|^{-1+\delta})$. Hence the theorem would still apply to a function of the form

$$f(z) = z^{-1+\delta}(c_0 + c_1 z + c_2 z^2 + \dots)$$
(59)

with a branch point at the origin.

In order to give an idea of the applicability of this theorem in practice we shall just mention—without proof—two properties of the resistivity transform, which takes the place of f when we want to calculate geoelectric sounding graphs:

1) It is analytic in the halfplane $\operatorname{Re}(\lambda) \ge 0$, thus admitting ω_0 to be—almost—equal to $\pi/2$.

2) The corresponding constant $K(\omega_0)$ can be proved to be less than 10 even for resistivity contrasts as high as 10⁴.

10. EXPLICIT EXPRESSION FOR THE SAMPLING ERROR

Substituting the exponential majorant (53) for $|\hat{F}(s)|$ in (52) we get

$$|G(v) - G^*(v)| \leq 4K(\omega_0) \cdot E(\omega_0, s_c, M)$$
(60)

where

$$E(\omega_0, s_c, M) = \int_{0}^{\infty} \exp\left(-2\pi\omega_0 s\right) (\mathbf{I} - \hat{P}(\Delta s)) \mathrm{d}s \qquad (61)$$

892

which can be evaluated directly by substituting (27) for $\hat{P}(\Delta s)$ and replacing the hyperbolic tangent functions by their expontential series expansion:

$$\tanh(z) = \mathbf{I} + 2\sum_{n=1}^{\infty} (-\mathbf{I})^n \exp(-2nz)$$
 (62)

After some manipulation we obtain

$$E(\omega_0, \Delta, M) = \frac{\exp(-2\pi\omega_0 s_c)}{2\pi M \sin(\omega_0/M)} + \frac{\omega_0}{\pi} \sum_{n=1}^{\infty} (-1)^n \frac{\exp(-n \cdot 2\pi^2 M s_c)}{(nM\pi)^2 - \omega_0^2}.$$
 (63)

We consider only $M \ge I$ and $\omega_0 < \pi$, so the summation is an alternating series of numerically decreasing terms. Hence the error from truncating this series has the same sign as and a smaller magnitude than the first term neglected, so when we discard the series altogether, we obtain the (strict) inequality

$$E(\omega_0, s_c, M) < \frac{\exp\left(-2\pi\omega_0 s_c\right)}{2\pi M \sin\left(\omega_0/M\right)}.$$
(64)

For large s_c or large M (or both) the magnitude of the series is negligible compared to the first term, so we may use the equality obtained from (64) by replacing the < sign by = as a good approximation to $E(\omega_0, s_c, M)$.

Large M correspond to a sharp cutoff of \hat{P} , and indeed

$$\lim_{M\to\infty} E(\omega_0, s_c, M) = \exp\left(-\frac{2\pi\omega_0 s_c}{2\pi\omega_0}\right), \tag{65}$$

which is the result one obtains by inserting a boxcar function in (6I).

Putting M equal to infinity in (64) gives the smallest error estimate, but the effect of decreasing M is not serious. If for instance $\omega_0 = \pi/2$, then Mequal to 3, 2, and I give error estimates which are only 5%, II% and 57% higher than the minimum value. Hence M = 2 is a good compromise, reducing the computation work to obtain H_{γ}^* without affecting the end result too much.

The exponential decay of the error with decreasing Δ yields good results even at moderate sampling densities. Let us consider the calculation of geoelectric sounding curves as a specific example. The apparent resistivity $\rho_a(r)$ is given by the formula

$$\rho_a(r) = \rho_1(\mathbf{I} + 2r^2 \int_0^{\infty} f(\lambda)\lambda \mathbf{J}_1(\lambda r) d\lambda) = \rho_1(\mathbf{I} + 2r^2 g(r))$$
(66)

where the characteristic function $f(\lambda)$ has the properties asserted in the preceeding paragraph. We actually calculate an approximation $rg^*(r) = G^*(\ln r)$ to $rg(r) = G(\ln r)$ so we get an approximate sounding curve

$$\rho_a^*(r) = \rho_1(\mathbf{I} + 2r^2 g^*(r)). \tag{67}$$

The relative error is

 $|\rho_{a}(r) - \rho_{a}^{*}(r)| / \rho_{1} = 2r |G(\ln r) - G^{*}(\ln r)| \leq 9.0 \cdot r \exp(-\pi^{2}s_{c})$ (68) where we have put $\omega_{0} = \pi/2$, K = 10 and M = 2 in (64).

With a sampling density of x points per decade we have $\Delta = (\ln 10)/x$. The maximum relative error for various values of x are presented in table 1. We see that a sampling rate of eight points per decade is quite sufficient, and that an increase in the computational work by 10% (to 9 samples per decade) yields an accuracy which is an order of magnitude better.

TABLE I

Maximum relative error in the calculation of geoelectric sounding curves for various sampling rates.

Samples per decade	3	4	5	6	7	8	9	10
max. rela- tive error	14.5	1.70	0.20	0.023	0.0027	3.2 ₁₀ -4	3.8 ₁₀ -5	4.410-6

II. TACKLING THE INFINITE SUMMATION

At this point—if not before—the pure mathematician would probably consider the problem solved. However, since our goal is to *actually* construct an *efficient* algorithm, we are obliged to show how to handle the infinite summation in (18). Due to the explicit expressions (31) and (33) for H_{ν}^{*} we can in fact apply a few tricks, as we shall see.

The input function f may have a branch point at the origin, as in (59), but let us for the sake of argument assume f to be analytic at the origin. Hence the power series expansion of f around the origin

$$f(\lambda) = c_0 + c_1 \lambda + c_2 \lambda^2 + \dots$$
(69)

will be convergent inside a disc extending out to the nearest singularity of f. In equidistant points $u = n\Delta$ we have

$$F(n\Delta) = \exp((-n\Delta) f(\exp((-n\Delta))) = \exp((-n\Delta)) \sum_{j=0}^{\infty} c_j \exp((-jn\Delta))$$
(70)

which will be rapidly convergent when n is large and positive. Usually it is quite easy to obtain explicit expressions for c_0 and c_1 for a given function f. In geoelectricity, for instance, we have

$$c_0 = (\rho_N - \rho_1)/\rho_1 \text{ and } c_1 = (R - \rho_N^2 C)/\rho_1,$$
 (71)

where $R = \sum_{j=1}^{N-1} \rho_j d_j$ and $C = \sum_{j=1}^{N-1} d_j / \rho_j$, ρ_j and d_j being the resistivity and thick-

ness of the *j*th layer. c_2 is more complicated, however, so we want to choose N^+ so large that the terms with $k \ge 2$ can be neglected when $n \ge N^+$. This is obviously justified if

$$|c_2/c_0| \cdot \exp\left(-2N^+\Delta\right) < \varepsilon_0 \tag{72}$$

where the computer's representation error ε_0 for real numbers is 10⁻¹⁰, say.

Hence we consider the sum

$$S_{N^{+}}^{+}(m) = \sum_{n=N^{+}}^{\infty} F(n\Delta) H_{\nu}^{*}((m-n)\Delta).$$
(73)

Writing $x = \exp(-\Delta) < I$ in the series expansion (70), substituting in (73) and changing the order of summation we get

$$S_{N}^{+}(m) = \sum_{j=0}^{\infty} c_{j} \chi^{(j+1)N^{+}} T_{j+1}(m-N^{+}), \qquad (74)$$

where the $T_j(m)$ are defined by

$$T_{j}(m) = \sum_{l=0}^{\infty} x^{jl} H_{\nu}^{*}((m-l)\Delta); \qquad j = 0, \text{ I} \dots$$
(75)

Note that these numbers depend only on H_{ν}^* and not on the input function F, so they need only be computed once and for all. But even this is not necessary, because they satisfy the simple recursion formula:

$$T_{j}(m+1) = H_{\nu}^{*}((m+1)\Delta) + x^{j}T_{j}(m)$$
(76)

so that it is enough to calculate them for *one* particular value of m, which we choose to be large and negative so that *all* of the values of H_{ν}^{*} appearing in (75) can be replaced by (31). Remembering (5) and (32) we have series expansions for the two terms in (31):

$$cH_{\nu}((m-l)\Delta) = \sum_{k=0}^{\infty} h_k x^{(\nu+1+2k)(l-m)}$$
(77)

where

$$h_{k} = c(-\mathbf{I})^{k} 2^{-(\nu+2k)} / (k! \Gamma(\nu+k+\mathbf{I})), \qquad (78)$$

and

$$\mathfrak{Fm}\{A_{\nu}((m-l)\Delta) \exp(i2\pi s_{c}(m-l)\Delta)\} = \sum_{k=0}^{\infty} a_{k} x^{(l-m)(k+1/2)/M}$$
(79)

where

$$a_k = (-\mathbf{I})^{l-m+1} M \pi \, \mathfrak{Fm}\{\hat{H}_{\nu}(s_c - \mathbf{i}(k+\mathbf{I}/2)/M\pi)\}. \tag{80}$$

Let us consider that part $T_j^{(1)}(m)$ of $T_j(m)$ which comes from the Bessel function term. Substituting (77) in (75) and changing the order of summation we get

$$T_{j}^{(1)}(m) = \sum_{k=0}^{\infty} h_{k} x^{-(\nu+1+2k)m} \sum_{l=0}^{\infty} x^{(\nu+1+j+2k)l}.$$
(81)

The rightmost summation is a geometric series and can be expressed in closed form. Hence

$$T_{j}^{(1)}(m) = \sum_{k=0}^{\infty} h_{k} x^{(-m)(\nu+1+2k)} / (1 - x^{\nu+1+j+2k}), \qquad (82)$$

which is rapidly convergent for negative *m*. For $T_j^{(2)}$ from the modulation term we obtain similarly

$$T_{j}^{(2)}(m) = \sum_{k=0}^{\infty} a_{k} x^{(-m)(k+1/2)/M} / (\mathbf{I} - x^{j+(k+1/2)/M}),$$
(83)

and we have

$$T_j(m) = T_j^{(1)}(m) + T_j^{(2)}(m).$$
 (84)

By means of the above formulas $S_{N^+}^+(m)$ can be calculated for all values of m with an accuracy which is only limited by the representation error in the computer.

The sum

$$S_{N-}^{-}(m) = \sum_{n=-\infty}^{N-} F(n\Delta) H_{\nu}^{*}((m-n)\Delta)$$
(85)

is connected to the behaviour of $f(\lambda)$ for large values of λ . If the response from a half-space with resistivity ρ_1 is treated separately by analytical means, so that f describes only deviations from this response due to layering, we find for both the geoelectric sounding and the electromagnetic dipole-dipole sounding case that $f(\lambda)$ is $O(\exp(-2d_1\lambda))$ for $\lambda \to \infty$. Hence $F(n\Delta)$ is $O(\exp(-n\Delta) \exp(-2d_1 \exp(-n\Delta)))$ for $n \to -\infty$, which solves our problem immediately, since this function goes abruptly to zero for negative n. If f is not majorized by a decaying exponential function for large λ it may still be possible to obtain an asymptotic expansion in inverse powers of λ

$$f(\lambda) \sim \lambda^{-\alpha} \sum_{j=1}^{\infty} d_j \lambda^{-j}; \ \alpha \ge 0, \ \lambda \to \infty$$
(86)

which permits a technique similar to the one developed for S^+ to be applied. This is particularly useful when we want to calculate the electromagnetic half-space response for the perpendicular loop configuration. An explicit solution for this case has been found by Wait (1955), but it turns out to be an

896

expression involving the modified Bessel functions I_0 , K_0 , I_1 and K_1 of complex argument. There exist of course routines to calculate these functions (see for instance Burrel 1972), but they are neither simpler nor faster nor more accurate than the present algorithm, which moreover needs only to be called once instead of the four calls to the modified Bessel functions.

12. The Fourier Transform as a Special Case

The results we have obtained are valid for all real values of ν greater than minus one, and in particular for $\nu~=~\pm~1/2.$ Due to

$$\sup_{sin} (\lambda r) = (\pi/2)^{1/2} (\lambda r)^{1/2} J_{\mp 1/2} (\lambda r) (87)$$

we can interprete the cosine and sine transforms as Hankel transforms

$$2\int_{0}^{\infty} f(x) \sin_{\sin} (2\pi sx) dx = r^{1/2} \int_{0}^{\infty} f_1(\lambda) \lambda J_{\pm 1/2}(\lambda r) d\lambda$$
(88)

where $r = s(2\pi)^{1/2}$; $\lambda = x(2\pi)^{1/2}$, and $f_1(\lambda) = f(\lambda/(2\pi)^{1/2})/\lambda^{1/2}$. In ordinary time series analysis, where the input data would be values of f measured at equidistant times, the present method could not be applied, but in theoretical model calculations we are free to choose a logarithmic sampling scheme for f, and under these circumstances this method has proved to be considerably more efficient than the Fast Fourier Transform (FFT). The reason can be seen as follows:

The behaviour of the transform $\hat{f}(s)$ for large values of s is mainly controlled by the behaviour of f(x) near the origin (see for instance Morse and Feshbach 1953, p. 462). In order to obtain $\hat{f}(s)$ with a given accuracy we must therefore represent f(x) with a certain minimum sampling density near the origin (sampling error). Once an adequate sampling density has been chosen we must (in case of the FFT) apply this sampling rate to an interval so large that f(x)can be neglected outside of it (aliasing error). With the present method the total number of sampling points needed to achieve the same accuracy is considerably smaller because they are placed strategically better.

A theoretical VLF problem has recently been studied by Olsson (1978). The numerical solution of an integral equation in the wave number domain had to be Fourier transformed to yield the response in the space domain. In order to achieve about four digits accuracy the FFT needed 512 sampling points, while the present method needed less than 100 points. Hence the size of the linear equation system could be reduced by more than a factor 25, which was quite important in this case. When better accuracy is wanted, the advantage of this method relative to the FFT is even more pronounced. A logarithmic sampling rate of 10 points per decade was used. Increasing this by only 10% to 11 points per decade would give one digit more right away, while a refined

FFT result would require 1024 sampling points, so that the equation system would contain more than one million array elements—a severe task even for computers of today.

13. ACKNOWLEDGEMENT

We want to thank dr P. Weidelt for pointing out early in 1975 that the Fourier transform (7) can be obtained *analytically* as in (8). This hint started the whole thing. Thanks are also due to prof. D. S. Parasnis and prof. S. Saxov for encouraging this mathematical digression, and to our colleagues for inspiring discussions. The typing was done by I. Lundmark, the drawings by Torben Riis, and proofreading of formulas by N. B. Christensen, whom we thank for their good work. The financial support from the Swedish Council for Technical Development (STU) to one of us, H. K. Johansen, is gratefully acknowledged.

Appendix A: Properties of $H_{\nu}(\zeta)$

$$\hat{H}_{\nu}(\zeta) = 2^{-i2\pi\zeta} \frac{\Gamma((\nu+1)/2 - i\pi\zeta)}{\Gamma((\nu+1)/2 + i\pi\zeta)}, \zeta = s + i\sigma.$$
(A1)

Because $I/\Gamma(\zeta)$ is an entire function, the poles and residues of $\hat{H}_{\nu}(\zeta)$ are obtained as

$$\zeta_n = -((\nu + I)/2 + n)/\pi, \quad n = 0, I, 2, \dots$$
 (A2)

$$R_n = (-1)^n 2^{-(2n+\nu+1)} i / (\pi \Gamma(\nu + n + 1) \Gamma(n + 1)).$$
 (A3)

The values of $\hat{H}_{\nu}(\zeta)$ with arguments placed symmetrically with respect to either the real axes or the imaginary axes are closely related, due to $\overline{\Gamma(\zeta)} = \Gamma(\overline{\zeta})$:

$$\hat{H}_{\nu}(s+i\sigma) = \overline{\hat{H}_{\nu}(-s+i\sigma)}$$
 (A4)

$$\widehat{H}_{\nu}(s+i\sigma) = \overline{[\widehat{H}_{\nu}(s-i\sigma)]^{-1}}.$$
(A5)

Another characteristic of $\hat{H}_{\nu}(\zeta)$ is a recursion formula, which is derived using $\Gamma(\zeta + \mathbf{I}) = \zeta \Gamma(\zeta)$:

$$\hat{H}_{\nu}(\zeta + i/\pi) = 4((\nu + I)/2 - i\pi\zeta) ((\nu + I)/2 + i\pi\zeta - I) \hat{H}_{\nu}(\zeta).$$
(A6)

The properties eq. (A4) and eq. (A5) are important for reducing computational work. Similarly eq. (A6)—in connection with the fact that asymptotic values of $\hat{H}_{\nu}(\zeta)$, $|\zeta| \to \infty$ can easily be calculated—yields a simple way to compute $\hat{H}_{\nu}(\zeta)$.

The following inequality is achieved from eq. (A6):

$$|\hat{H}_{\nu}(\zeta + ni/\pi)| / (2\pi |s|)^{2n} \ge |\hat{H}_{\nu}(\zeta)|, \qquad |s| > 0.$$
(A7)

This inequality is used for estimating an upper limit for $|\hat{H}_{\nu}(\zeta)|$ in the lower halfplane.

In the upper halfplane

$$|\hat{H}_{\nu}(\xi)| \leq 2^{2\pi\delta} \exp(\pi^2 |s|) \Gamma^2((\mathbf{I} + |\nu|)/2 + \pi\delta)/\pi$$

$$|\nu| > -\mathbf{I}, \delta > (\mathbf{I} + |\nu|)/2\pi$$
(A8)

is valid. Eq. (A8) is obtained mainly by means of $\Gamma(\zeta)\Gamma(I-\zeta) = \pi/\sin \pi \zeta$.

Appendix B: Properties of the Interpolating Function P

Let us consider functions P(v) whose Fourier transform $\hat{P}(s)$ can be expressed in the following form:

$$\hat{P}(s) = \hat{O}(s + s_0 + 1) - \hat{O}(s + s_0).$$
 (B1)

By means of mathematical induction we obtain that

$$\sum_{n=0}^{N} \hat{P}(s+n) = \hat{O}(s+s_0+(N+1)) - O(s+s_0)$$
(B2)

and

$$\sum_{n=-1}^{-N} \hat{P}(s+n) = \hat{O}(s+s_0) - \hat{O}(s+s_0-N).$$
(B3)

Combining (B2) and (B3) and letting $N \to \infty$ we derive

$$\sum_{-\infty}^{\infty} \hat{P}(s+n) = \lim_{N \to \infty} (\hat{O}(N) - \hat{O}(-N)),$$
(B4)

provided that $\hat{O}(s)$ is finite, and $\hat{O}(N)$ and $\hat{O}(-N)$ tend to a finite limit for $N \to \infty$. Hence

$$\sum_{n \neq 0} \hat{P}(s+n) = \lim_{N \to \infty} (\hat{O}(N) - \hat{O}(-N)) - \hat{P}(s).$$
(B5)

This property (B5) is very important for estimating the aliasing error using P(v) as interpolating function.

We consider

$$\begin{split} \Delta \hat{P}(\Delta \zeta) &= \Delta (\mathbf{I} - \exp\left(-2\pi/a\right)) \left[\mathbf{I} + \exp\left((\Delta \zeta - \mathbf{I}/2)2\pi/a\right]^{-1} \cdot \\ &\cdot \left[\mathbf{I} + \exp\left(-(\Delta \zeta + \mathbf{I}/2)2\pi/a\right)\right]^{-1}, \ \zeta &= s + \mathrm{i}\sigma. \end{split} \tag{B6}$$

The poles and residues are achieved from a Taylor expansion around the zeroes of each denominator:

$$\zeta_k^{\pm} = i(k+1/2)a/\Delta \pm 1/2\Delta; \quad k = ..., -2, -1, 0, 1, 2, ...,$$
(B7)

$$R_k^{\pm} = \mp a/2\pi\Delta. \tag{B8}$$

The asymptotic behaviour follows directly from (B6), when large values of s are inserted:

$$|\Delta \hat{P}(\Delta \zeta)| \sim \Delta(\mathbf{I} - \exp(-2\pi a)) \exp(-2\pi \Delta |s|/a), |s| \to \infty.$$
 (B9)

It is easy to show that

$$\hat{P}(s) = 1/2 \tanh ((s+1/2)\pi/a) - 1/2 \tanh ((s-1/2)\pi/a).$$
 (B10)

Defining

$$\widehat{O}(s) = 1/2 \tanh(s\pi/a) \tag{B11}$$

we have

$$\hat{P}(s) = \hat{O}(s - 1/2 + 1) - \hat{O}(s - 1/2)$$
 (B12)

Therefore, eq. (B5) is valid for $\hat{P}(s)$ defined by (B10):

$$\sum_{n \neq 0} \hat{P}(s+n) = \mathbf{I} - \hat{P}(s).$$
(B13)

We observe that (B12) may be written as a convolution:

$$\hat{P} = 2\hat{O} * I_I; \tag{B14}$$

where $i \cdot I_I(s) = i/2(\delta(s + 1/2) - \delta(s - 1/2))$ is the Fourier transform of sin (πu) (Bracewell 1965 p. 79). \hat{O} defined by (B11) is the Fourier transform of $ia/2 \sinh(\pi a u)$ (Bracewell 1965 p. 366), so by the convolution theorem we obtain

$$P(u) = \frac{a \cdot \sin(\pi u)}{\sinh(\pi a u)} \equiv \sinh(u)$$
(B15)

For |u| small we have sinh $(\pi au) \approx \pi au$ and hence

$$P(u) \approx \frac{a \sin (\pi u)}{\pi a u} = \operatorname{sinc} (u); |\pi a u| \ll 1$$
(B16)

while sinh $(\pi au) \approx \operatorname{sgn}(u) \cdot \exp(-\pi a \mid u \mid)/2$ for $\mid u \mid$ large, so that

$$P(u) \approx (a/2) \exp(-\pi a | u |) \sin(\pi | u |); |\pi a u | \gg 1.$$
 (B17)

The main difference between sinc and sinsh is thus that the latter is damped exponentially for large arguments, which is due to the analyticity of its Fourier transform.

Appendix C: Properties of $Q(\zeta)$

We consider $\hat{Q}(\zeta)$ defined by

$$\widehat{Q}(\zeta) = (\mathbf{I} + \exp\left(-\mathrm{i}(\zeta - \mathrm{i}\sigma_c)2\pi/b\right))^{-1}, \zeta = s + \mathrm{i}\sigma.$$
(CI)

Using a Taylor expansion around the zeroes of the denominator we obtain the poles and residues of $\hat{Q}(\zeta)$:

$$\zeta_n = (n + 1/2)b + i\sigma_c; n = ..., -2, -1, 0, 1, 2, ...,$$
(C2)

and

$$R_n = b/2\pi i. \tag{C3}$$

The asymptotic behaviour in the upper halfplane for $\sigma \rightarrow \infty$ is trivial:

$$|Q(\zeta)| \sim \exp(-(\sigma - \sigma_c)2\pi/b), \sigma \to \infty.$$
 (C4)

Similarly, we have in the lower halfplane

$$|Q(\zeta)| \sim \mathbf{I}, \, \sigma \to -\infty. \tag{C5}$$

References

- ABRAMOWITZ, A., and STEGUN, I., 1965, Handbook of Mathematical Functions, Dover Publications, Inc., New York.
- BARANOV, W., 1976, Calcul des courbes de sondages électriques à l'aide fonctions d'échantillonnage, Geoph. Prosp. 24, 222-232.
- BERNABINI, M., and CARDARELLI, E., 1978, The use of filtered Bessel functions in direct interpretation of geoelectrical soundings, Geophysical Prospecting 26, 841-852.
- BICHARA, M., and LAKSHAMANAN, J., 1976, Fast automatic processing of resistivity soundings, Geoph. Prosp. 24, 354-370.
- BRACEWELL, R., 1965, The Fourier transform and its applications, McGraw-Hill Inc.
- DAS, U. C., and GHOSH, D. P., 1974, The determination of filter coefficients for the computation of standard curves for dipole resistivity sounding over layered earth by linear digital filtering, Geoph. Prosp. 22, 765-780.
- DAS, U. C., GHOSH, D. P., and BIEWINGA, D. T., 1974, Transformation of dipole resistivity sounding measurements over layered earth by linear digital filtering, Geoph. Prosp. 22, 476-489.
- GHOSH, D. P., 1970, The application of linear filter theory to the direct interpretation of geoelectrical resistivity measurements. (Ph.D. Thesis)
- GHOSH, D. P., 1971, The application of linear filter theory to the direct interpretation of geoelectrical resistivity sounding measurements, Geoph. Prosp. 19, 192-217.
- GHOSH, D. P., 1971a, Inverse filter coefficients for the computation of apparent resistivity standard curves for a horizontally stratified earth, Geoph. Prosp. 19, 769-775.
- JOHANSEN, H. K., 1975, An interactive computer/graphic-display-terminal system for interpretation of resistivity soundings. Geoph. Prosp. 23, 449-458.
- KOEFOED, O., 1972, A note on the linear filter method of interpreting resistivity sounding data, Geoph. Prosp. 20, 403-405.
- KOEFOED, O., GHOSH, D. P., and POLMAN, G. J., 1972, Computation of type curves for electromagnetic depth sounding with a horizontal transmitting coil by means of a digital linear filter, Geoph. Prosp. 20, 406-420.
- KOEFOED, O., 1976, Error propagation and uncertainty in the interpretation of resistivity sounding data, Geoph. Prosp. 24, 31-48.
- KOEFOED, O., and DIRKS, F. J., 1979, Determination of resistivity sounding filters by the Wiener Hopf least square method, Geoph. Prosp. 27, 245-250.
- MORSE, P. M., and FESHBACH, H., 1953, Methods of theoretical physics, McGraw-Hill Inc.
- OLSSON, O., 1978, Scattering of electromagnetic waves by a perfectly conducting halfplane below a stratified overburden, Radio Science 13, 2, 391-397.
- PALEY, R. E. A. C., and WIENER, N., 1934, Fourier transforms in the complex domain, American Mathematical Society.
- VERMA, R. K., and KOEFOED, O., 1973, A note on the linear filter method of computing electromagnetic sounding curves, Geoph. Prosp. 21, 70-76.